THE UNIVERSITY OF ALBERTA

ON SOME PLASMA INSTABILITIES

by

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A THESIS

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ABSTRACT

A general review on the theory of plasma instabilities in a discharge tube is given for the helical type instabilities (azimuthal wave numbers m=1,2). The major effect causing the steady state radial density profile, as calculated by previous authors, has been the boundary. They have considered the walls as an infinite sink, thereby causing a density gradient. After a suggestion by Glicksman we consider the self induced azimuthal magnetic field as the major cause of the radial density profile. The effect of the walls has been reduced by demanding that the tube radius be large.

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CHAPTER I

INTRODUCTION

The motion of charged particles in the positive column of a discharge tube has been studied both theoretically and experimentally in great detail by applying various configurations of electric and magnetic fields.

If a plasma is contained in a long cylindrical tube in which the walls are considered to be places of recombination, the electrons due to their lighter mass will tend to diffuse to the walls at a much faster rate than the ions. However, a charge separation will result, and this charge separation will set up an electric field which will retard the electron diffusion and at the same time enhance the diffusion of the positive ions. The net result is that the electrons and positive ions will tend to diffuse together at an intermediate rate. This is called ambipolar diffusion. If a constant electric field is applied parallel to the axis of the tube the diffusion perpendicular to the electric field will remain ambipolar. However, the motion of the particles along the electric field will be controlled by the electric field.

The motion of a charged particle in a constant magnetic field, with a constant electric field applied parallel to the magnetic field, may be described as the sum of a gyration around the magnetic field lines, a drift motion across the lines and a motion parallel to the field. In an ionized gas (plasma) this motion of particles has to be modified by particle interactions, such as collisions and phenomena caused by space charges. In a cylinder in which the magnetic and electric fields are parallel to the axis of the tube the motion of the particles parallel to the electric field remains unaltered by the addition of a constant magnetic field applied parallel to the electric field. However, the flow of particles perpendicular to the magnetic and electric field lines is inhibited. By applying a magnetic field parallel to the tube axis the ambipolar diffusion coefficient is reduced by $(1 + \mu_- \mu_+ B_{oz}^2)^{-1}$, where B_{oz} is the applied magnetic field along the axis and μ_- and μ_+ are the electron and ion mobilities respectively.

However, it has been pointed out by Bohm, Burhop, Massey and Williams[1] that random fluctuations of charge density and plasma oscillations may produce electric fields which in their turn give rise to drift motions across magnetic field lines. This drift motion provides an additional mechanism for ionized matter to

move across a magnetic field and may, when it dominates over collision diffusion, introduce considerable difficulties into the physics of a plasma in a magnetic field. Bohm and collaborators performed experiments with a plasma in a magnetic field and came to the conclusion that the diffusion was not consistent with collision phenomena. The diffusion across the lines was much greater than predicted by collision theory and hence an additional mechanism must be present.

Lehnert[2] studied the diffusion processes in a helium discharge in a longitudinal magnetic field, by using a long discharge tube.

The results of Lehnert's experiments showed that the radial diffusion agreed with collision theory up to a certain critical value of the magnetic field. Above this critical magnetic field an instability was found to occur and the results indicated a diffusion rate much greater than that given by collision theory. This gave support of the additional mechanism which supported the enhanced diffusion as stated by Bohm et al.

Kadomtsev and Nedospasov[3] showed how this effect discovered by Lehnert could be explained by the fact that a screw type disturbance will grow in the presence of a large enough magnetic field due to the

interaction between the current and the magnetic field. In the absence of a magnetic field any disturbance of the plasma will be rapidly terminated by the increased diffusion of particles from regions of excess density. In particular, the twisting and wriggling of the current channel causes an increase in the flux of particles striking the wall at those places where the channel comes close to it and a decrease in the flux on the reverse side. This has the effect of restoring the original state of the plasma. In the presence of a magnetic field there is an additional force, the interaction of current and magnetic field, acting on the plasma. With a helical distortion of the plasma this force acts either towards the axis of the discharge or towards the walls according to whether the helix is right or left handed. When the force acts in the direction of the walls there is a build-up of the initial disturbance. As the magnetic field is increased this force becomes greater and so does the transverse diffusion, and along with it, the stabilized action of the walls is reduced. Thus for sufficiently large magnetic fields the plasma column becomes unstable. Using this explanation Kadomtsev and Nedospasov were able to theoretically derive a relation between the longitudinal electric and magnetic fields which gave

a diffusion coefficient which agreed with the results obtained by Lehnert up to and beyond the critical magnetic field.

THE INSTABILITY MECHANISM

A physical model of the growth mechanism of the helical wave was developed by Hoh and Lehnert[4]. The argument is begun by assuming a small, quasi-neutral, screw-shaped perturbation of positive and negative carriers to be superimposed on the steady state unperturbed distribution. The general form for the perturbation in cylindrical coordinates would be

$$F(r) \exp[i(m\phi + kz - \omega t)]$$

with $F(r)$ determined by the boundary conditions at the wall, ϕ being the azimuthal angles, k the wave number, ω the frequency of the oscillation. The perturbation is shown schematically in Fig. (1a) for m and k having opposite signs (a right-handed helix). The applied longitudinal electric field E_{oz} tends to lift up the ion screw relative to the electron screw. This separation is equivalent to a rotation of one screw relative to the other, as shown in Fig. (1a) and in the cross-sectional view [Fig. (1b)]. The charge separation resulting from the screw rotation creates an azimuthal

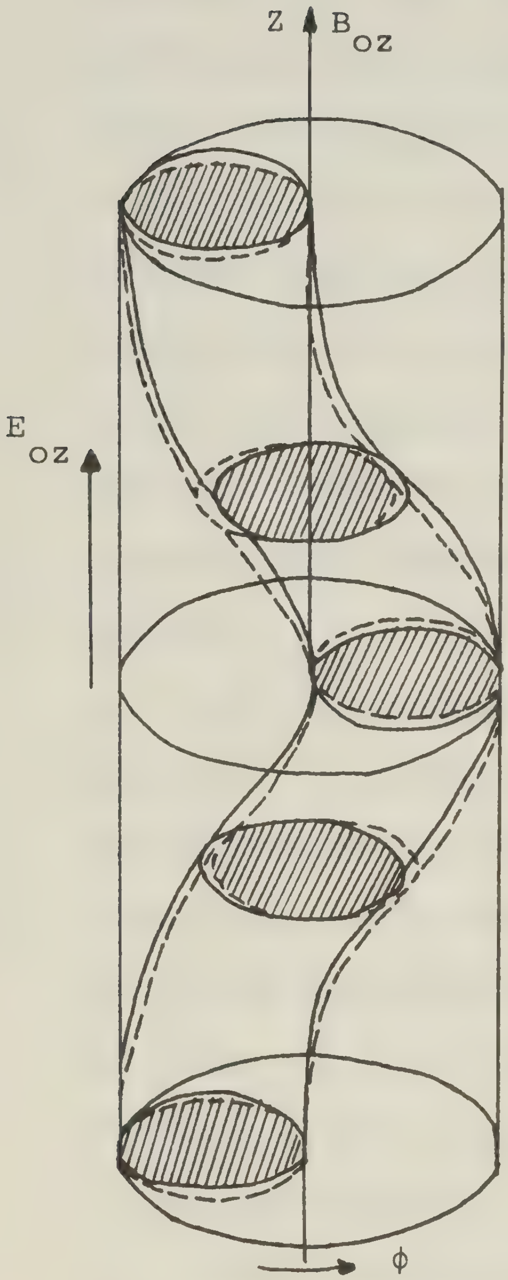

Fig. (1a)

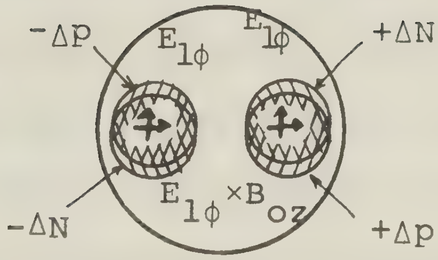

Fig. (1b)

Fig. (1)
Right handed helical perturbation for $|m=1|$. (a) Perturbed-electron distribution indicated by dashed lines and perturbed-ion distribution by solid lines. (b) Cross-sectional view showing charge separation and resulting perturbed electric field $E_{1\phi}$. Directions of carrier flow due to the combined effect of $E_{1\phi}$ and B_{oz} are indicated by vectors $\hat{r}E_{1\phi}B_{oz}$.

electric field E_{ϕ}'. The azimuthal electric field together with the longitudinal magnetic field B_{oz}, act on the unperturbed distribution to produce a radial flow of particles. The direction of the radial flow of particles is determined by the vector product $\hat{r}.(\vec{E} \times \vec{B}) = E_{\phi}'B_{oz}$. As can be seen from figure (1b) the perturbed azimuthal electric field causes the particles to drift to the wall at the peak perturbed density and pulls them to the center at the minimum perturbed density, thus enhancing the perturbation.

On the other hand, the ambipolar electric field E_{or} which is positive will act together with the longitudinal magnetic field to produce a radial flow of particles in the negative azimuthal direction. The resulting ambipolar electric field will therefore have the effect of reducing the charge separation created by the axial electric field $\vec{E}_{oz}$, making the column more stable.

The effect of a self induced azimuthal magnetic field $B_{\phi}(r)$ on the column will be to reduce the outward radial flow of particles caused by the azimuthal electric field E_{ϕ}'. This is because the induced azimuthal magnetic field will act with the applied axial electric field to cause a flow of particles in

towards the axis of the cylinder. The final flow of particles in the radial direction is given by the term

$$[\hat{r}.(\vec{E} \times \vec{B}(r))] = E'_{\phi}B_{oz} - E_{oz}B_{\phi}(r) .$$

Instability occurs in the column when the charge separation created by the axial electric field is great enough to overcome the dissipative effects caused by the ambipolar electric field and the recombination of particles inside the bulk. It may therefore be expected that for a given value of the axial electric field E_{oz} there will be a threshold value of the axial magnetic field, B_{oz} critical, above which the growth of the helix is observed.

CHAPTER II

HISTORICAL SURVEY

II.1 Description of the system

The system to be studied consists of an electron-ion plasma situated in parallel electric and magnetic fields. The plasma is generated in a cylindrical discharge tube and we assume that the plasma is ionized weakly enough such that we have to only take into account collisions between charged particles and neutrals. We assume that our plasma obeys the perfect-gas law, and that although the density of the plasma may vary throughout the tube, the density of positive and negative particles is the same at every point. We assume that internal electric fields may exist in the plasma although charge neutrality is assumed. This assumption of quasineutrality is reasonable since only small differences in the densities can produce very strong internal electric fields.

The apparatus consists of a long cylindrical discharge tube and the length of the tube is much greater than the diameter of the inner radius, the inner radius of the tube being of the order of 1 cm while the tube length is greater than 4 meters, so that end effects can be neglected. The electric field is produced by an electric discharge in the tube and

the magnetic field is produced by a magnetic coil. The longitudinal electric field is measured with an electrostatic voltmeter connected between two probes about one meter apart in the positive column of the tube.

II.2 m = 1 Instability

The helical plasma density wave (instabilities) was first proposed by Kadomtsev and Nedospasov as an explanation for the anomalous diffusion observed experimentally by Lehnert for a plasma in a longitudinal magnetic field. In solving the problem Kadomtsev and Nedospasov started from two basic sets of equations. The first set of equations is the equation of continuity for ions and electrons. The second set of equations is an equation of motion for ions and electrons. They assume that the frequency of the collisions between the ions and the neutral gas molecules is much greater than the collisions between the electrons and the neutral gas and also much greater than the cyclotron frequency and the frequency of the oscillations of the disturbance. Therefore in the equation of motion for the positive ions they are able to neglect the effect of the magnetic field.

The equation of continuity is

$$\frac{\partial n}{\partial t} + \nabla . \vec{\Gamma}_{\mp} = n\xi \quad , \tag{II.1}$$

where "-" refers to electrons and "+" refers to ions. $\vec{\Gamma}_{\mp} = n\vec{V}_{\mp}$ is the particle flux vector, $\vec{V}_{\mp}$ being the average velocity of either component of the plasma, and ξ is the number of ion-electron pairs created per second per electron. The equation of motion which relates the flux to the forces is for electrons

$$\vec{\Gamma}_{-} + D_{-}\nabla n + \mu_{-}n\vec{E} + \mu_{-}\vec{\Gamma}_{-} \times \vec{B} = 0 \quad . \tag{II.2}$$

For the ions the corresponding equation is

$$\vec{\Gamma}_{+} - \mu_{+}n\vec{E} = 0 \quad . \tag{II.3}$$

In equations (II.2) and (II.3), D_{-} is the electron diffusion coefficient, $\mu_{\mp}$ the electron and ion mobilities and $\vec{B}$ and $\vec{E}$ are the magnetic and electric fields respectively.

Kadomtsev and Nedospasov then solve equations (II.2) and (II.3) for $\vec{\Gamma}_{-}$ and $\vec{\Gamma}_{+}$ explicitly, take their divergences and combine the results with equation (II.1) to obtain the two following equations in which $\vec{\Gamma}_{-}$ and $\vec{\Gamma}_{+}$ have been eliminated.

For the electrons:

$$- \frac{\partial n}{\partial t} + n\xi + D_{-}'\nabla^{2}n + \mu_{-}' \nabla.(n\vec{E}) + \mu_{-}\mu_{-}'\nabla.(n(\vec{B}\times\vec{E}))$$

$$+ \mu_{-}^{2}D_{-}'\vec{B}.\nabla(\vec{B}.\nabla n + n\vec{B}.\vec{E}) = 0 \quad . \tag{II.4}$$

For the ions:

$$- \frac{\partial n}{\partial t} - \mu_{+}^{'} \nabla . (n\vec{E}) + n\xi = 0 \qquad \text{(II.5)}$$

where

$$\mu_{\mp}^{'} = \frac{\mu_{\mp}}{1 + \mu_{\mp}^{2} B_{oz}^{2}} \;, \quad D_{-}^{'} = \frac{D_{-}}{1 + \mu_{-}^{2} B_{oz}^{2}} \;. \qquad \text{(II.6)}$$

In obtaining equations (II.4) and (II.5) Kadomtsev and Nedospasov used the conditions that $\nabla . \vec{B} = 0$. They also assumed that the internal magnetic field is negligible relative to the large uniform longitudinal magnetic field which is being externally applied to the column and therefore $\nabla \times \vec{B} = 0$.

a) <u>Steady-state solution</u>

By using equations (II.4) and (II.5) they solve for the case of an unperturbed cylindrical plasma in a uniform longitudinal magnetic field, where there are no gradients of density or electric fields in the axial and azimuthal directions.

The solution they obtain for the density is

$$n_o(r) = N_o J_o(\beta_o r) \;, \qquad \text{(II.7)}$$

where $\beta_o = \alpha_o / R$, where α_o is the first root of the zero order Bessel function J_o, R is the radius of the tube, N_o is the density along the axis of the tube.

They then determine a value for the ambipolar electric field by requiring that the radial flux of ions is equal to the radial flux of electrons

$$\vec{\Gamma}^{r}_{-} = \vec{\Gamma}^{r}_{+} \quad . \tag{II.8}$$

The value they obtain for the radial electric field is

$$E_{or} = - \frac{D_{-}}{\mu_{-} + \mu_{+}(\mu_{-}^{2}B_{o}^{2})} \frac{1}{n_{o}} \frac{dn_{o}}{dr} \quad . \tag{II.9}$$

Now

$$\frac{dn_{o}}{dr} = - \beta_{o} J_{1}(\beta_{o} r) \quad . \tag{II.10}$$

The radial electric field becomes

$$E_{or} = \frac{D_{-}}{\mu_{-} + \mu_{+}(\mu_{-} B_{oz})^{2}} \beta_{o} \frac{J_{1}(\beta_{o} r)}{J_{o}(\beta_{o} r)} \quad . \tag{II.11}$$

The radial electric field is positive, that is it points from the cylinder axis towards the walls of the tube. This is what you would expect of the ambipolar electric field. Also since $J_1(\beta_o r)$ is zero at $r = 0$, along cylinder axis, the value of the radial electric field is also zero on the tube axis. For symmetry reasons this is what you expect.

b) Time-dependent solution

The authors then solve for the conditions under which the plasma becomes unstable, that is the conditions under which the anomalous diffusion as observed by Lehnert will be dominant in the tube. To solve the problem the authors have chosen to consider a small perturbation in their calculated steady state values for the density and space charge potential. The form of the perturbation they choose is

$$f(r) \exp[i(m\phi + kz - \omega t)] \ , \qquad \text{(II.12)}$$

where ϕ is the azimuthal angle, k is the wave number, ω is the frequency of the oscillation. Both ϕ and k are real numbers while ω is a complex number.

The density and electric fields can be written as

$$\begin{aligned} n &= n_o + n_1 \\ \vec{E} &= \vec{E}_o + \vec{E}_1 \ , \end{aligned} \qquad \text{(II.13)}$$

where n_o and $\vec{E}_o$ are the steady state values of density and electric field, n_1 and $\vec{E}_1$ are the perturbed values of the density and electric fields. Equations (II.13) are then substituted for n and $\vec{E}$ in equations (II.4) and (II.5) and only terms of the lowest order in the

perturbation are kept. Equations (II.4) and (II.5) become:

For electrons

$$n_1\xi - \frac{\partial n_1}{\partial t} + D_-^{'}\nabla^2 n_1 + \mu_-^{'}\nabla.(n_o\vec{E}_1 + n_1\vec{E}_o)$$

$$+ \mu_-\mu_-^{'}\nabla.[\vec{B} \times (n_o\vec{E}_1 + n_1\vec{E}_o)] + \mu_-^2 D_-^{'} B.\nabla(\vec{B}.\nabla n_1)$$

$$+ \mu_-^2\mu_-^{'}\vec{B}.\nabla[\vec{B}.(n_o\vec{E}_1 + n_1\vec{E}_o)] = 0 \qquad \text{(II.14)}$$

For ions

$$n_1\xi - \frac{\partial n_1}{\partial t} - \mu_+^{'}\nabla.(n_o\vec{E}_1 + n_1\vec{E}_o) = 0 \;. \qquad \text{(II.15)}$$

Equations (II.14) and (II.15) contain only the perturbation terms and do not contain any steady state terms which have been grouped together and equated equal to zero separately.

n_1 and $\vec{E}_1$ are written explicitly as

$$n_1 = f(r)\,\exp[i(kz + m\phi - \omega t)]$$

$$V_1 = g(r)\,\exp[i(kz + m\phi - \omega t)] \;, \qquad \text{(II.16)}$$

where

$$\vec{E}_1 = -\,\text{grad}\,V_1 \;. \qquad \text{(II.17)}$$

Since we demand that the density be zero at the tube walls, $f(r=R) = 0$.

They define a new variable

$$g(r) = \frac{\ell(r)}{n_o(r)} \quad . \tag{II.18}$$

Since $g(R)$, which is essentially the amplitude of the potential of the electric field at the boundary, must be finite and since $n(R)$ has finite values, due to the boundary conditions at the walls, $\ell(R)$ must then be finite. Using condition (II.18) they substitute equations (II.16) into equations (II.14) and (II.15) obtaining:

For electrons

$$D_-' \frac{1}{r} \frac{d}{dr} \left(r \frac{df}{dr}\right) + \mu_-' \frac{1}{r} \frac{d}{dr} (rE_{or} f) - \Big((i\omega - \xi) - ik\mu_- E_{oz}$$

$$+ D_- k^2 + \frac{D_-' m^2}{r^2} - i\mu_- \mu_-' E_{or} \frac{mB_{oz}}{r}\Big) f$$

$$= \mu_-' \frac{1}{r} \frac{d}{dr} \left(r \frac{d\ell}{dr}\right) - \mu_-' \frac{1}{r} \frac{d}{dr} (\ell r p)$$

$$+ \left(- \frac{\mu_-' m^2}{r^2} - \mu_- k^2 - i\mu_- \mu_-' \frac{mB_{oz} p}{r}\right) \ell$$

$$= 0 \tag{II.19}$$

For ions

$$-\mu_+' \frac{1}{r}\frac{d}{dr}(rE_{or}f) - ((i\omega-\xi) + ik\mu_+E_{oz})f$$

$$= -\mu_+' \frac{1}{r}\frac{d}{dr}\left(r\frac{d\ell}{dr}\right) + \mu_+' \frac{1}{r}\frac{d}{dr}(\ell rp)$$

$$+ \left(+\frac{\mu_+' m^2}{r^2}\right) + \mu_+ k^2)\ell$$

$$= 0 \quad , \tag{II.20}$$

where

$$p(r) = \frac{1}{n_o}\frac{dn_o}{dr} \quad . \tag{II.21}$$

The authors then assume that the radial dependence of the perturbation has the following form:

$$\begin{aligned} n_1 &= n' J_1(\beta_o r) \\ V_1 &= V' J_1(\beta_1 r) \ , \end{aligned} \tag{II.22}$$

where $\beta_1 = (\alpha_1/R)$, where α_1 is the first root of the first order Bessel function, n' and V' are constants independent of the radial positions. Each of equations (II.19) and (II.20) contains a real and imaginary part which can be separately equated equal to zero. Equations (II.19) and (II.20) contain five parameters, the wave number k, the frequency of the oscillation ω, the azimuthal mode m, the applied

electric field E_{oz} and the applied magnetic field B_{oz}. Using the conditions that the imaginary part of the frequency must be greater than zero and that the derivative of the frequency with respect to the wave number must also be zero, they are able to derive a relation between the critical electric and critical magnetic field for the m = 1 mode.

The reason for demanding that the imaginary part of the frequency be equal to zero can be shown by the following calculation. The condition that the column becomes unstable is that the helix grows with time. Therefore if we look at the time dependent portion of the instability the real part must be greater than 1 for the helix to grow. The time dependent portion of the helical instability is given by

$$\exp[-i\omega t] \quad . \tag{II.23}$$

Now

$$\omega = \omega_r + i\omega_i \quad . \tag{II.24}$$

Putting (II.24) into (II.23) we get

$$\exp[-i\omega_r t] \, \exp[\omega_i t] \quad . \tag{II.25}$$

We can see from relation (II.25) that the condition that the wave grows is that $\exp[\omega_i t]$ is positive. Therefore ω_i must be greater than zero for the helix

to grow. If $\omega_i = 0$ this will define the boundary between the region for which the helix grows with time and decays with time and hence the boundary between the stable and unstable regions. All points on this boundary would be critical points.

Their calculated theoretical critical points agreed with the corresponding experimental critical points obtained by Lehnert[2].

Johnson and Jerde[5] extended the theory of Kadomtsev and Nedospasov in that the diffusion and magnetic field interaction terms are included in the ion equation of motion, and also in that the equations are solved in a more rigorous manner without any prior assumptions regarding the form of the radial dependence of the perturbations of density and potential.

The equation of motion for the ions used by Johnson and Gerde is

$$\vec{\Gamma}_+ = -\mu_+ n\vec{E} + D_+\nabla n - (\vec{\Gamma}_+ \times \vec{B})\ \mu_+ \ . \qquad \text{(II.26)}$$

Equation (II.26) differs from equation (II.3) in that it includes the diffusion term for the ions $D_+\nabla n$ and the magnetic field interaction term $\mu_+(\vec{\Gamma}_+ \times \vec{B})$.

Following the same procedure as Kadomtsev and Nedospasov, Johnson and Jerde used equation (II.26) to solve for $\vec{\Gamma}_+$ and then substituted the value into

the equation of continuity for ions equation (II.1). The equation obtained is

$$- \frac{\partial n}{\partial t} + n\xi - \mu_+^{'}\nabla.(n\vec{E}) + D_+^{'}\nabla^2 n + \mu_+\mu_+^{'}\nabla.(n(\vec{B}\times\vec{E}))$$

$$+ \mu_+^2 D_+^{'}\vec{B}.\nabla(\vec{B}.\nabla n - n\vec{B}.\vec{E}) = 0 \quad . \tag{II.27}$$

Equation (II.27) differs from equation (II.5) in that it includes the diffusion term and two terms due to the magnetic field interaction.

Eliminating the radial electric field between equations (II.27) and (II.4) they obtain an equation for the density of the unperturbed plasma. The equation they obtain is

$$\frac{1}{r}\frac{d}{dr}(r\frac{dn_o}{dr}) + \beta_o^2 n_o = 0 \quad . \tag{II.28}$$

The solution to this well known equation is a Bessel function of order zero

$$n = N_o J_o(\beta_o r) \quad , \tag{II.29}$$

where

$$\beta_o^2 = \frac{(\mu_+^{'} + \mu_-^{'})\xi}{\mu_+^{'}D_-^{'} + \mu_-^{'}D_+^{'}} \quad , \tag{II.30}$$

and N_o is the density along the cylinder axis.

The only difference between equation (II.29) and equation (II.7) is the definition of the first zero β_o. This occurs as a result of the inclusion of the diffusion and magnetic field terms into the theory for ions and results in new coefficient for n_o being formed when E_{or} is eliminated between equations (II.27) and (II.4).

They then calculated the radial electric field using condition (II.8). The value obtained is

$$E_{or} = \frac{D_+' - D_-'}{\mu_+' + \mu_-'} \frac{1}{n_o} \frac{dn_o}{dr} . \tag{II.31}$$

Using equation (II.29) this can be rewritten as

$$E_{or} = \frac{D_-' - D_+'}{\mu_+' + \mu_-'} \beta_o \frac{J_1(\beta_o r)}{J_o(\beta_o r)} . \tag{II.32}$$

The radial electric field (II.32) differs from that of Kadomtsev and Nedospasov only through the mobility and diffusion coefficient terms and in the value of the first zero of the zero order Bessel function. It can be seen by comparing (II.25) and (II.11) that the diffusion coefficient for the ions is now present whereas it was not present in equation (II.11).

Equation (II.25) can be rewritten as

$$E_{or} = \frac{(D_- - D_+) - \mu_+\mu_- B_{oz}^2 \frac{k}{e} (\mu_- T_+ - \mu_+ T_-)}{(\mu_+ + \mu_-) + \mu_+\mu_- B_{oz}^2 (\mu_+ + \mu_-)} \left(\frac{\beta_o J_1(\beta_o r)}{J_o(\beta_o r)}\right) \tag{II.33}$$

It can be seen by looking at equation (II.33) that E_{or} is positive provided the relation

$$D_- - D_+ > \mu_+\mu_- B_{oz} \frac{k}{e} (\mu_- T_+ - \mu_+ T_-) \qquad \text{(II.34)}$$

holds true.

However, it would seem that for large values of B_{oz} the relation (II.34) would not be true, therefore the radial electric field would be negative and point in towards the cylinder axis. This observation was not pointed out by Johnson and Jerde.

Johnson and Jerde solved for the condition for the onset of the helical instability using the same perturbation method as Kadomtsev and Nedospasov. The inclusion of the diffusion and magnetic field terms in the equation of motion for the ions resulted in a different equation of continuity for the ions. That is in replace of equation (II.20) they used

$$\begin{aligned} - \mu_+' \frac{1}{r} \frac{d}{dr} (rE_{or} f) - ((i\omega - \xi) + ik\mu_+ E_{oz}) f \\ + D_+' \frac{1}{r} \frac{d}{dr} (r \frac{df}{dr}) + (D_+ k^2 + \frac{D_+'}{r^2} - i\mu_+\mu_+' \frac{E_{or}B_{oz}}{r}) f \\ = -\mu_+' \frac{1}{r} \frac{d}{dr} (r \frac{d\ell}{dr}) + \mu_+' \frac{1}{r} \frac{d}{dr} (\ell r p) \\ + (\frac{\mu_+'}{r^2} + \mu_+ k^2) \ell - i\mu_+\mu_+' \frac{B_{oz} p \ell}{r} \quad . \qquad \text{(II.35)} \end{aligned}$$

However, Johnson and Jerde did not make any assumptions regarding the radial form of the perturbed density $n_1(r)$ and perturbed potential $V_1(r)$ as did Kadomtsev and Nedospasov. Instead they made Hankel transformation of the complete equations of continuity of the perturbed ions and electrons, equations (II.35) and (II.19) respectively. The independent variable r was transformed into α_j, where $J_m(\alpha_j r/R) = 0$, where R is the tube radius. $n_1(r)$ and $V_1(r)$ were transformed into the functions $F(\alpha_j)$ and $L(\alpha_j)$. They then were able to eliminate $F(\alpha_j)$ from the system of equations and the remaining system of algebraic equations could be written as

$$L(\alpha_j) + \sum_{i=1}^{\infty} E_{ij} L(\alpha_j) = 0 \quad , \qquad \text{(II.35a)}$$

where the matrix elements E_{ij} are known functions of axial magnetic field and constants. Due to the extreme complexity of E_{ij} only the first term in the matrix was used

$$1 + E_{11} = 0 \ . \qquad \text{(II.35b)}$$

Equations (II.35b) were solved, using the same conditions as Kadomtsev and Nedospasov, for the critical electric field in terms of the critical magnetic field. They then put these points on a graph (fig. 2). One can see

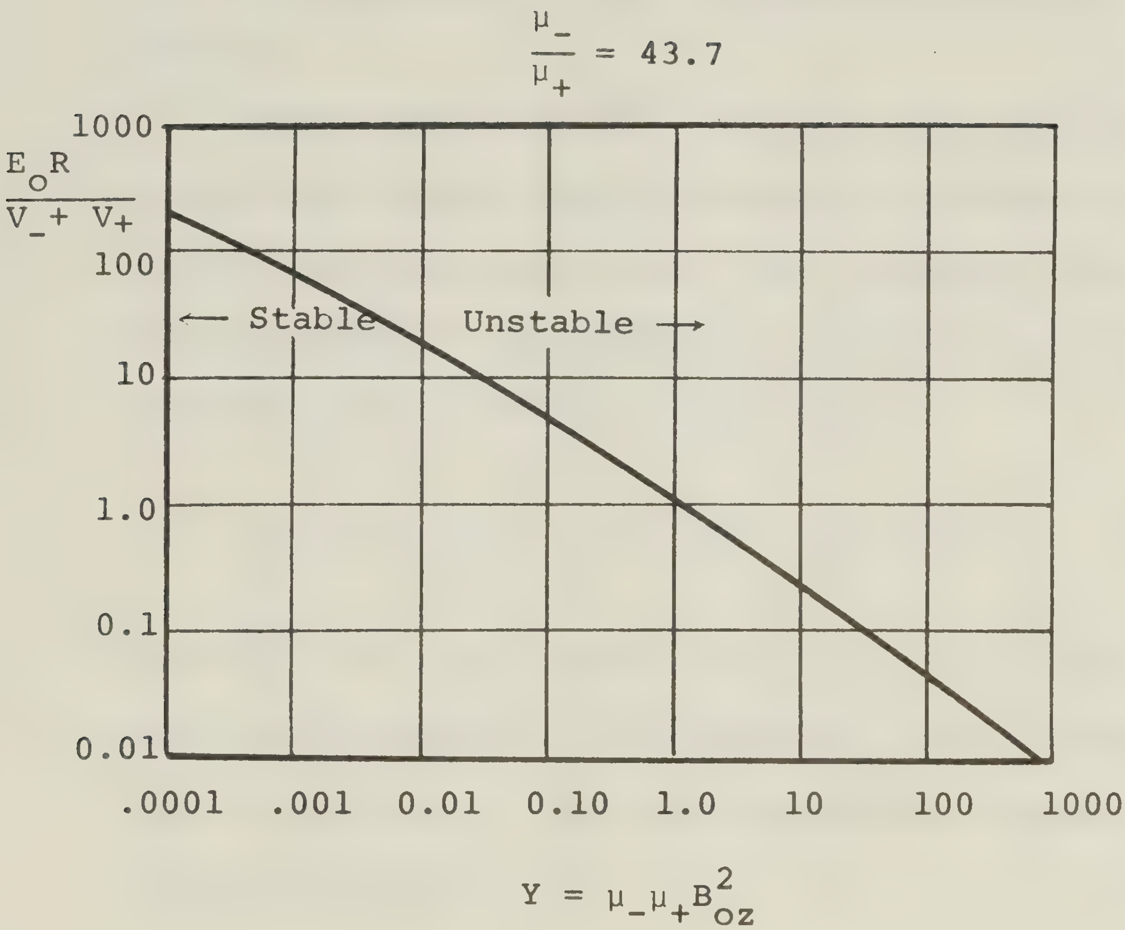

Fig. (2). Stability criterion for the helical mode in helium as a function of the longitudinal magnetic field.

by looking at the resulting curve that the larger the applied axial electric field the smaller is the required axial magnetic field necessary to make the instability grow. This is consistent with the theory for the mechanism of the instability as pointed out earlier.

Paranjape et al.[6] took the same approach as Johnson and Jerde, but modified the problem by considering the effect of a small self induced azimuthal magnetic field. The azimuthal magnetic field is simply related to the electric current by

$$B_\phi(r) = \frac{e}{c^2}\frac{1}{r}\int_0^1 n(r')(V_z^+ - V_z^-)\, r'dr' \;, \qquad \text{(III.36)}$$

where $V_z^{\mp}$ are the z components of the drift velocities which they assume to be independent of the radial and axial directions. They also defined an average plasma density given by

$$\int_0^r n(r')\; r'dr' = \frac{\bar{n}r^2}{2} \quad . \qquad \text{(III.37)}$$

Therefore the azimuthal magnetic field can be approximately written as

$$B_\phi(r) = \frac{eV_z\bar{n}r}{c^2} \quad , \qquad \text{(III.38)}$$

where

$$V_z = V_z^+ - V_z^- \quad . \qquad \text{(III.39)}$$

Following the same procedure as Johnson and Jerde they obtained an equation for the steady state plasma. The equation they obtained is:

$$\frac{d^2N}{d\rho^2} + \left(\frac{1}{\rho} + \frac{A\rho}{2}\right)\frac{dN}{d\rho} + (A + B)N = 0 \ , \qquad \text{(III.40)}$$

where

$$\rho = \frac{r}{R} \ ; \qquad N = \frac{n_o(r)}{\bar{n}} \qquad 2T = T_+ + T_- \qquad \text{(III.41)}$$

$$A = \frac{V_z^2 \bar{n} e^2 R^2}{c^2 k_B T} = \frac{eB_\phi(R)(\mu_- + \mu_+)}{k_B T} E_{oz}R \qquad \text{(III.42)}$$

$$B = \frac{eR^2\xi}{2\mu'' k_B T} \quad \text{where} \quad \mu'' = \frac{\mu_- - \mu_+}{\mu_+ + \mu_-}(1 + \mu_-\mu_+ B_{oz}^2)^{-1} \ . \qquad \text{(III.43)}$$

B can be rewritten as

$$B = \frac{R^2\xi(\mu_+' + \mu_-')}{\mu_+'D_-' + \mu_-'D_+'} \ . \qquad \text{(III.44)}$$

Therefore B can be written as

$$B = R^2\beta_o^2 \ ,$$

where β_o^2 is defined by Johnson and Jerde.[5] If in equation (II.40) we put the azimuthal magnetic field term equal to zero (A = 0), the solution we obtain is

$$N = N_o J_o(\gamma_o \rho) \quad , \tag{III.45}$$

where

$$\gamma_o = \frac{(\mu_+' + \mu_-')}{\mu_+' D_-' + \mu_-' D_+'} \quad . \tag{III.46}$$

N_o is the density at the origin. This can be re-written as

$$n = n_o J_o(\beta_o r) \; ,$$

which is the same solution as Johnson and Jerde: this is what you would expect to obtain.

Using the method of perturbation the next approximation to the solution of equation (II.40) they obtain is

$$N = N_o[J_o(\gamma_o\rho) + a_1 J_o(\gamma_1\rho)] \; , \tag{III.47}$$

where $\gamma_1 = 5.52$ is the second zero of the Bessel function of order zero and a_1 is a small number. The term $a_1 J_o(\gamma_1 \rho)$ increases the density at the center from $N_o J_o(\gamma_o \rho)$ and decreases the density near the boundary wall.

The radial component of the internal electric field was determined by requiring that the radial flux of electrons is equal to the radial flux of ions. The resulting radial electric field they obtained is

$$E_r = \frac{D_+' - D_-'}{\mu_+' + \mu_-'} \frac{1}{n_o} \frac{dn_o}{dr} - \frac{(\mu_- - \mu_+)B\, E_{oz}}{1 + \mu_+\mu_- B_{oz}^2} \quad . \qquad \text{(III.48)}$$

It can be seen from equation (III.48) that if we put the azimuthal magnetic field equal to zero $B_\phi = 0$ they obtain the same value for the radial electric field as Johnson and Jerde. It can also been seen from equation (III.48) the effect of B_ϕ is to reduce the radial electric field. This is what one would expect since the induced magnetic field increases the density towards the center of the bulk and reduces the density for the larger radial values. Therefore the flow of electrons to the wall is decreased and hence a smaller radial electric field is required to keep the flow of ions to the wall equal to the flow of electrons.

Using the same method as Kadomtsev and Nedospasov they derived a relation between the critical electric and critical magnetic fields for various strengths of the self induced azimuthal magnetic field. They then plotted curves of the critical electric field against the critical magnetic field for various strengths of the induced magnetic field (fig. (3)). It can be seen from the graph that for increasing values of the induced magnetic field increasing values of the applied axial magnetic field is required to cause the instability

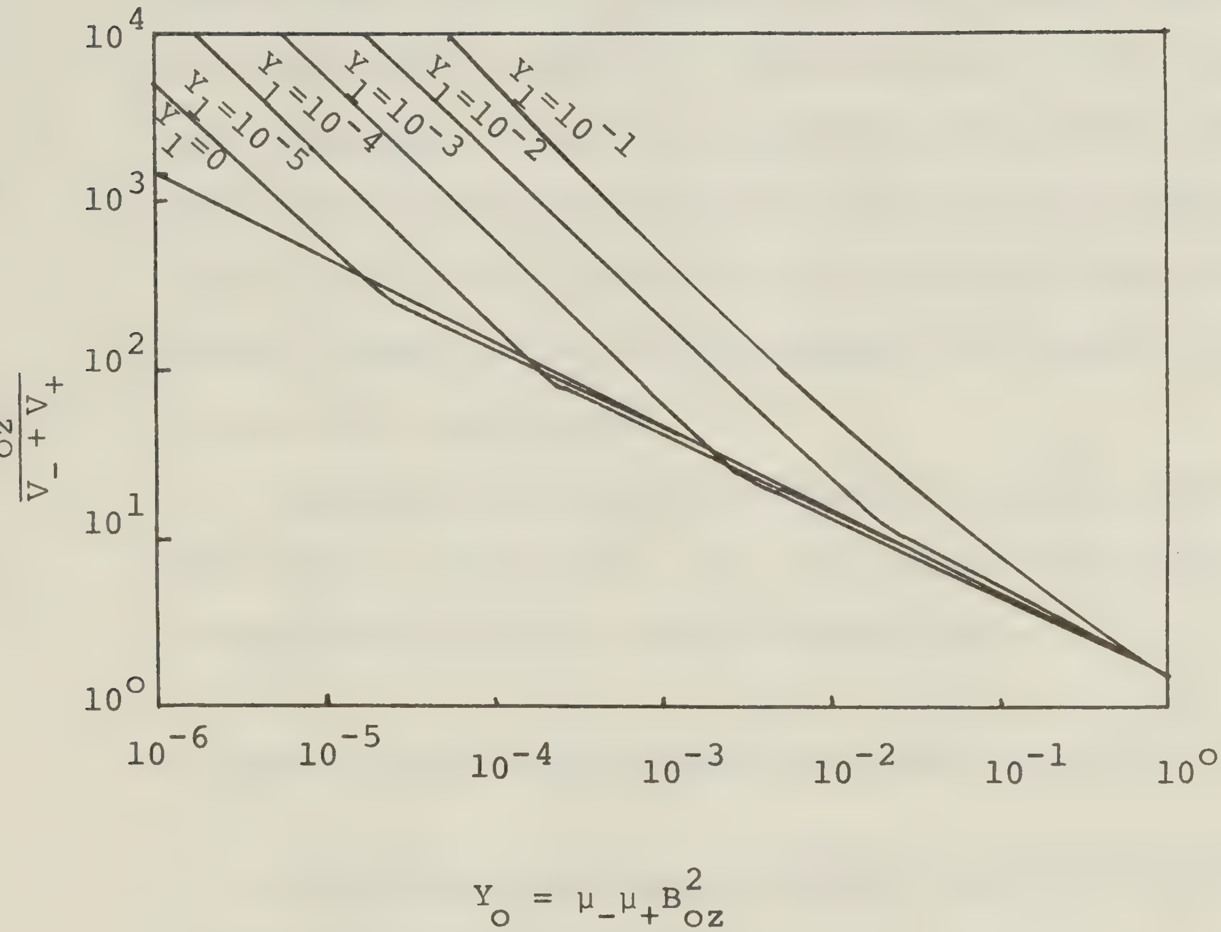

Fig. (3). Stability criterion for the helical mode in helium $(\mu_-/\mu_+) = 43.7$, where $Y_1 = (\mu_-\mu_+B_\phi^2(R)$.

to grow for a given critical electric field. Therefore the effect of the self induced azimuthal magnetic field is to make the plasma more stable.

c) <u>m = 1 Instability Finite Amplitude</u>

Holter and Johnson[7] have expanded the theory as developed by Johnson and Jerde by superimposing a finite amplitude helix upon the steady state density and potential instead of a perturbation of the steady state density. They first calculate the functional form for the density of the background plasma including the effect of the finite amplitude helix on the background plasma. They then determine the stability conditions for the plasma.

When solving for the background density, steady state case, they assume that the density and space charge potential have the following form

$$n = \mathrm{Re}\ \{N_o h_o(r) + N_1 f(r)\exp[i(\omega t + kz + m\phi)]\}.$$

h_o is the background plasma density which is affected by the amplitude of the helix.

$$V = \mathrm{Re}\ \{V_o(r) + V_1 g(r)\exp[i(\omega t + kz + m\phi + \delta)]\}, \quad \text{(III.49)}$$

where N_o is the density along the cylindrical axis, N_1 is the finite amplitude of the helix. V_o is the space

charge potential of the background plasma which is affected by the finite amplitude of the helix, V_1 is the amplitude of the helical potential. δ is the phase shift between the helical density and the helical potential and is a real number. Equations (II.49) are then substituted into the equation of continuity for the electrons equation (II.4) and for ions equation (II.27), keeping all terms including the non-linear terms which are terms with coefficient products N_1V_1. The resulting equation obtained is

$$[h_o\xi + D_{\mp}^{'}\nabla_r^2 h_o \pm \mu_{\mp}^{'}\nabla_r(h_o E_{or})]N_o + [\mp \mu_{\mp}^{'} \frac{1}{2} \cos\delta\nabla_r(\frac{fdg}{dr})$$

$$- \mu_{\mp}^{'}\mu_{\mp}B_{oz}m \frac{1}{2} \sin \delta\nabla_r(\frac{fg}{r})]N_1V_1 \exp(-2\omega_i t)$$

$$+ \mathrm{Re}(\{ [f\xi - i\omega f + D_{\mp}^{'}\nabla_r^2 f - D_{\mp}^{'} \frac{m^2}{r^2} f \pm \nabla_r(fE_{or})$$

$$+ \frac{im}{r} \mu_{\mp}^{'}\mu_{\mp}B_{oz}E_{or}f - D_{\mp}k^2 f \pm i\mu_{\mp}kE_{oz}f]N_1$$

$$+ [\mp \mu_{\mp}^{'}\nabla_r(h_o \frac{dg}{dr}) \pm \mu_{\mp}^{'} \frac{m^2}{r^2} h_o g + im\mu_{\mp}^{'}\mu_{\mp}B_o\nabla_r(\frac{h_o g}{r})$$

$$\pm \mu_{\mp}k^2 h_o g - \frac{im}{r} \mu_{\mp}^{'}\mu_{\mp}B_{oz}h_o \frac{dg}{dr}]N_oV_1 \exp(i\delta)\}.$$

$$.\exp[i(\omega t + kz + m\psi)]) + \mathrm{Re}(\{ [\mp \mu_{\mp}^{'}\nabla_r(f \frac{dg}{dr})$$

$$+ 2m^2 \mu_{\mp}^{'} \frac{fg}{r^2} - 2i\mu_{\mp}^{'}\mu_{\mp}B_{oz} \frac{m}{r} \nabla_r(\frac{fg}{r}) \pm 2\mu_{\mp}k^2 fg].$$

$$.\exp[i(2(\omega t + kz + m\psi) + \delta)]\} \frac{1}{2} N_1V_1\exp(-2\omega_i t))=0 \quad \text{(III.50)}$$

Equation (II.50) differs from the corresponding equation as derived by Johnson and Jerde in that they include the generation of higher harmonics from the first harmonic term. This can be seen by looking at the terms which multiply

$$\exp[i(2(\omega t + kz + m\phi) + \delta)]N_1 V_1 \ .$$

It can also be seen from looking at equation (II.50) that the inclusion of these higher order terms affects the zero harmonic equation. This can be seen by looking at the two terms which multiply $N_1 V_1$. It should be pointed out that the non-linear terms or second harmonic terms in equations (II.50) are incomplete since they do not include the higher harmonic terms in the definition of their density and space charge potential equations (II.49). This omission implies that they do not incorporate in their theory the generations of higher harmonics by the finite amplitude first harmonic.

To obtain the steady state solution they look at the zero harmonic portion of the equation setting it equal to zero and assuming that $\omega_i t << 0$. The zero harmonic portion of the equation becomes assuming $m = 1$

$$[h_o \xi + D'_{\mp} \nabla_r^2 h_o \pm \mu'_{\mp} \nabla_r (h_o E_{or})]N_o + [\mp \mu'_{\mp} \frac{1}{2} \cos\delta \nabla_r (f \frac{dg}{dr})$$
$$- \mu'_{\mp} \mu_{\mp} B_{oz} \ \frac{1}{2} \sin\delta \nabla_r (\frac{fg}{r})]N_1 V_1 = 0 \ . \qquad \text{(III.51)}$$

To solve for the density h_o they first eliminate E_{or} between the ion and electron equation in (III.51) and obtain the following equation

$$\nabla_r^2 h_o + \beta_o^2 h_o - \phi \nabla_r \left(\frac{fg}{r}\right) = 0 \ , \qquad \text{(II.52)}$$

where

$$\beta_o^2 = \frac{(\mu_+' + \mu_-')\xi}{\mu_+' D_-' + \mu_-' D_+'}$$

$$\phi = \frac{1}{2} \ \frac{N_1 V_1 \left[\left(\frac{\mu_-}{\mu_+}\right)^{\frac{1}{2}} + \left(\frac{\mu_+}{\mu_-}\right)^{\frac{1}{2}}\right] \mu_- \mu_+ B_{oz}^2 \ im\delta}{N_o (V_- + V_+)} \ . \qquad \text{(II.53)}$$

Equation (II.52) differs from the corresponding equation in the theory of Johnson and Jerde, equation (II.28), in that it contains the term

$$- \phi \nabla_r \left(\frac{fg}{r}\right) \ .$$

One can see immediately by looking at equation (II.53) that this term arises from the inclusion of the non-linear terms in their equation (II.50). This term was neglected by Johnson and Jerde since they considered their helical density distribution to be perturbation. If you put $\phi = 0$ you obtain equation (II.28) which is what one would expect.

Holter and Johnson solved equation (II.52) by writing it in integral form and using certain relations between Bessel functions. They also used the boundary conditions that the density be one at the origin ($h_o(0) = 1$) and zero at the tube wall $h_o(R) = 0$. The equation they obtained for the density h_o is

$$h_o(r) = J_o(\beta_o r) + \frac{1}{2}\pi\phi[J_o(\beta_o r) + \int_0^{\beta_o r} \frac{J_1^2[(\beta_1/\beta_o)\beta_o r']}{h_o}$$

$$\times Y_o'(\beta_o r') - Y_o(\beta_o r) \int_0^{\beta_o r} \frac{J_1^2[(\beta_1/\beta_o)\beta_o r']}{h_o}$$

$$\times J_o(\beta_o r')\beta_o dr] \quad . \tag{II.54}$$

Equation (II.54) is a non-linear integral equation for h_o and was solved by Holter and Johnson using a digital computer. It can be seen by looking at equation (II.54), if ϕ is put equal to zero, which is the same a neglecting non-linear terms, we get the well known density solution which is a zero order Bessel function.

Holter and Johnson plotted a graph of relative density against radial location for various positive values of ϕ (fig. (4)). It can be seen from their graph that for ϕ equal to zero the density profile is that of a zero order Bessel function. As ϕ becomes very large, the density profile is given approximately

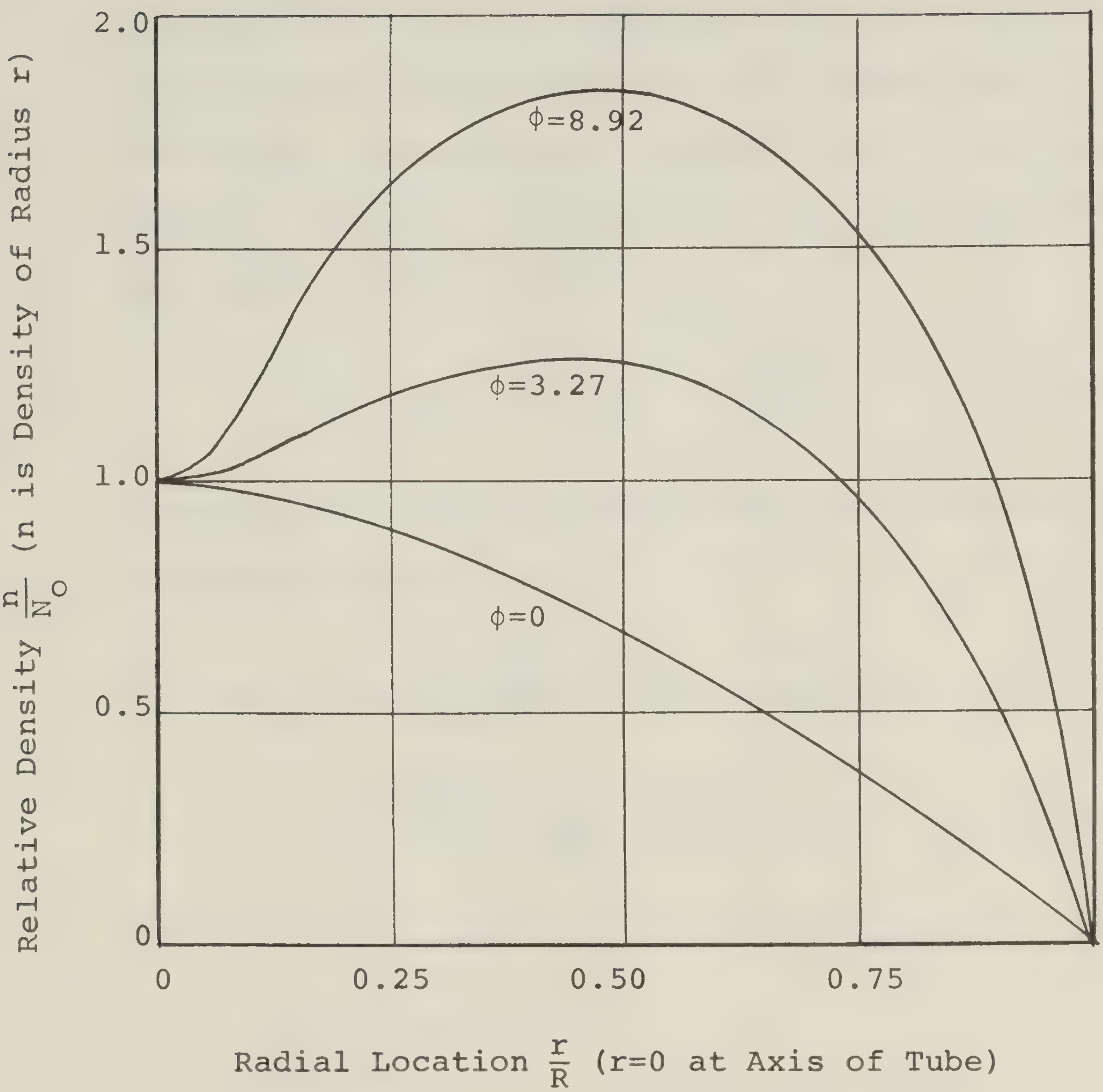

Fig. (4). Radial density profile of the positive column for ϕ=0, 3.27 and 8.92.

by a first order Bessel function (i.e. $J_1(\beta_1 r)$) with a deviation occurring near the origin.

Holter and Johnson then calculate the radial electric field not by requiring that the radial flux of electrons and ions are equal as Johnson and Jerde did. Instead they eliminated the ξ term from equation (II.51). They assume the following form for the radial portion of helix

$$f(r) = J_1(\beta_1 r) \qquad g(r) = \frac{J_1(\beta_1 r)}{h_o(r)} \quad . \tag{II.55}$$

The resulting equation they obtain for the radial electric field is

$$E_{or} = \alpha\beta_o \frac{1}{h_o}\frac{dh_o}{dr} + \frac{\alpha\beta_o}{\eta}\left[\phi \frac{J_1^2(\beta_1 r)}{\beta_o r h_o} + \psi \frac{J_1^2(\beta_1 r)}{h_o}\frac{d}{dx_o}\left(\frac{J_1(\beta_1 r)}{h_o}\right)\right] , \tag{II.56}$$

where $Y_o = \beta_o r$, where $\alpha\beta_o$ can be written as

$$\frac{D_+' - D_-'}{\mu_+' + \mu_-'} ,$$

$$\psi = \frac{\phi}{a_7 \tan\delta} ,$$

$$a_7 = \left[\left(\frac{\mu_-}{\mu_+}\right)^{1/2} - \left(\frac{\mu_+}{\mu_-}\right)^{1/2}\right] \frac{(\mu_-\mu_+B_{oz}^2)^{1/2}}{(1 + \mu_-\mu_+B_{oz}^2)} ,$$

$$\eta = (1 + \mu_- \mu_+ B_{oz}^2) \frac{(\mu_- V_- - \mu_+ V_+)}{(\mu_- - \mu_+)(V_- + v_+)} - \mu_- \mu_+ B_{oz}^2 \quad .$$

$V_{\mp}$ are the terminal energies for the ions and electrons.
$V_{\mp} = \frac{k_B T_{\mp}}{e}$, where $T_{\mp}$ are the ion and electron temperatures.

Looking at equation (II.56) it can be seen that if we put $\phi = 0$ the solution for the radial electric field is the same as that obtained by Johnson and Jerde.

Holter and Johnson using the first harmonic portion of equation (II.50) and following much the same procedure as Johnson and Jerde obtained a relation between the critical electric and critical magnetic fields in terms of the parameter ϕ. It can be seen from fig. (5) that for a given critical electric field the larger the value of ϕ (that is the larger the ampli- of the helix) the larger is the critical magnetic field required to make the plasma unstable.

II.3a <u>m = 2 Instability</u>

Holter and Johnson[8] extended their own theory by investigating the conditions whereby the helical instability (m = 1 mode) could excite higher modes in the plasma column. They investigated under what conditions on m = 2 mode can be excited in the presence of a finite amplitude helical mode (m = 1). They

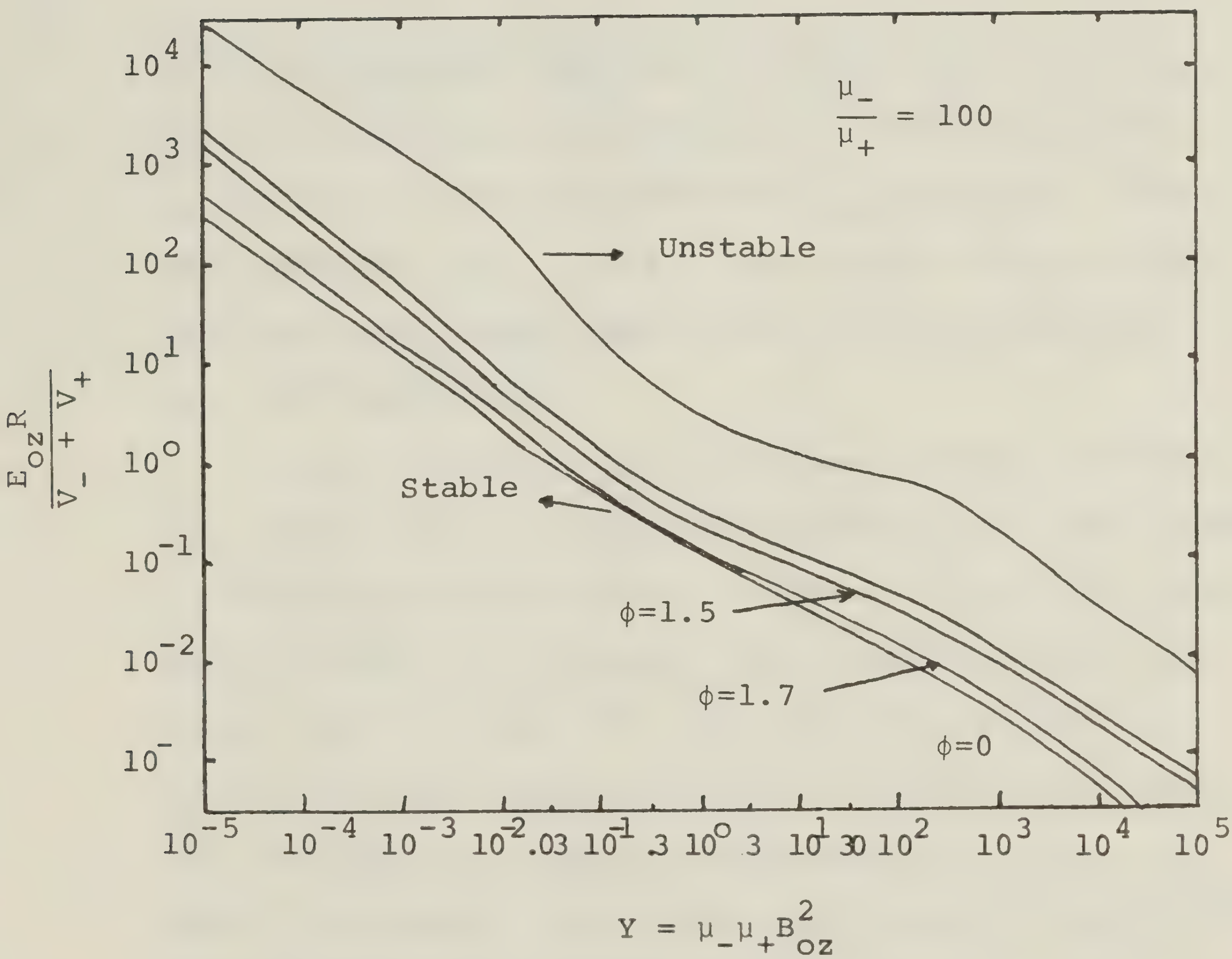

Fig.(5). Stability criterion for the finite m = 1 instability as a function of the longitudinal magnetic field for different values of the amplitude factor ϕ.

consider the m = 2 mode to be a perturbation of the m = 1 mode. They express the density n and space charge potential V in the form

$$n = \mathrm{Re}\{N_o h_o(r) + \sum_{m=1}^{2} N_m f_m(r) \exp[i(\omega_m t + k_m z + m\phi)]\}$$

$$V = \mathrm{Re}\{V_o(r) = \sum_{m=1}^{2} V_m g_m(r) \exp[i(\omega_m t + k_m z + m\phi + \delta_m)]\} \ . \tag{II.57}$$

As in the previous theories the axial wave number k_m and the azimuthal wave number m and the phase angle δ_m are real, while the frequency ω_m is complex. The amplitudes N_o, V_o, N_1 and V_1 are finite and N_2 and V_2 are considered small quantities such that products N_2V_2 are negligible.

They calculated the background density profile and radial electric fields and obtained the same values as they obtained previously. This is reasonable since they consider the terms N_2V_2 are considered small and hence the m = 2 mode will not affect the background density or space charge potential in the lowest order. They then derive stability conditions for the m = 2 mode, and compare their calculated critical fields with experimental observations.

II.3b Finite m = 2 Instability

Daugherty and Ventrice[9] extending the theory of Holter and Johnson[8] determined the radial form of the particle density by considering the effect of a finite m = 2 mode on the previously calculated background density, which had used a finite m = 1 mode. They use the same form for the density n and the space charge potential V as in equation (II.57) except that they consider the term N_2 and V_2 to be finite. They also assume the radial form of the helix to be

$$f_m(r) = J_m(\beta_m r)$$
$$g_m(r) = \frac{J_m(\beta_m r)}{h_o(r)} \ , \qquad \text{(II.58)}$$

where $h_o(r)$ is the steady state density and will be affected by the inclusion of a finite m = 2 mode. Substituting equations (II.57) into the equation of continuity for electrons, equation (II.4), and for ions equation (II.27), they obtain an equation which corresponds to equation (II.50). This equation differs from equation (II.50) in that it contains terms of order N_2V_2. The second harmonic equation is complete and also partial third and fourth harmonic terms are included. Following the same procedure as Holter

and Johnson[7] they eliminate the radial electric field from the zero harmonic equation and the resulting equation obtained is

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}h_o\right) + \beta_o^2 n_o - \frac{\phi_1}{r}\frac{d}{dr}\left(\frac{J_1^2(\beta_1 r)}{h_o}\right)$$

$$- \frac{\phi_2}{r}\frac{d}{dr}\left(\frac{J_2^2(\beta_2 r)}{h_o}\right) = 0 \tag{II.59}$$

$$\phi_1 = \frac{1}{2}(\mu_- + \mu_+)\, B \sin\delta_1 \frac{N_1 V_1}{N_o(V_- + V_+)}$$

$$\phi_2 = (\mu_- + \mu_+)\, B \sin\delta_2 \frac{N_2 V_2}{N_o(V_- + V_+)} \quad . \tag{II.60}$$

The only difference between equation (II.59) and the corresponding equation in the theory of Holter and Johnson[7], equation (II.52), is the additional term

$$- \frac{\phi_2}{r}\frac{d}{dr}\left(\frac{J_2^2(\beta_2 r)}{h_o}\right) \quad . \tag{II.61}$$

Looking at the definition of ϕ_2 from (II.60) it is clear that expression (II.61) arises as a result of the inclusion of the finite m = 2 mode in the density and space charge potential. This is not included in the theory of Holter and Johnson. So by putting $\phi_2 = 0$ we get the identical equation as (II.52). It should be pointed out that $\phi_1 = \phi$ when m is put equal to 1 in our definition of ϕ.

By requiring that $h_o(0) = 1$ and $h_o(R) = 0$ the authors were able to express equation (II.59) in the integral form as

$$h_o = J_o(x_o) + \phi_1 \frac{\pi}{2} [J_o(x_o) \int_0^{x_o} \frac{J_1^2[(\beta_1/\beta_o)x_o']}{h_o} Y'(x_o)dx_o'$$

$$- Y_o(x_o) \int_0^{x_o} \frac{J_1^2[(\beta_1/\beta_o)x_o']}{h_o} J_o'(x_o')dx_o']$$

$$+ \phi_2 \frac{\pi}{2} [J_o(x_o) \int_0^{x_o} \frac{J_2^2[(\beta_2/\beta_o)x_o'}{h_o} Y_o'(x_o')dx_o'$$

$$- Y_o(x_o) \int_0^{x_o} \frac{J_2^2[(\beta_2/\beta_o)x_o^2]}{h_o} J_o'(x_o')dx_o'] \ , \qquad \text{(II.62)}$$

where $x_o = \beta_o r$.

Looking at equation (II.62) one can see that if you put $\phi_2 = 0$ which is neglecting the N_2V_2 terms, or considering the m = 2 mode as small, one gets the solution as obtained by Holter and Johnson[7] equation (II.54). If you put both ϕ_1 and ϕ_2 equal to zero, that is to consider the m = 1 mode a perturbation of the background density, you get the solution of Johnson and Jerde equation (II.29). Finally, if you put ϕ_1 equal to zero and consider ϕ_2 as being finite you obtain the case of a finite m = 2 mode occurring without the presence of a m=1 mode.

Equation (II.62) was solved with a digital computer by Daugherty and Ventrice and the results obtained were plotted on four separate graphs showing the density against the radial position in the tube for various values of ϕ_1 and ϕ_2.

The first graph (fig. (6)) shows the density radial profile when ϕ_2 is put equal to zero for various values of ϕ_1. The curves obtained are the same as those obtained by Holter and Johnson[7] in fig. (4). This is what one would expect since putting $\phi_2 = 0$ reduces equation (II.62) to that of equation (II.54). Looking at the curves in fig. (6) we can see that for ϕ_1 equal to zero the radial form for the density is a zero order Bessel function and as ϕ_1 is increased the radial density function approaches a first order Bessel function, with a deviation occurring near the origin. As ϕ_1 becomes large there is an off axis peak occurring at approximately $r/R = .46$ with no dip occurring near the origin.

In the second graph (fig. (7)) they show the case of ϕ_1 equal to zero and ϕ_2 varying. This is the case of a pure $m = 2$ mode occurring without the presence of a $m = 1$ mode in the column. They found that as ϕ_2 became large the radial density profile approached a second order Bessel function with a deviation occurring near the origin. For large values of ϕ_2 an off axis

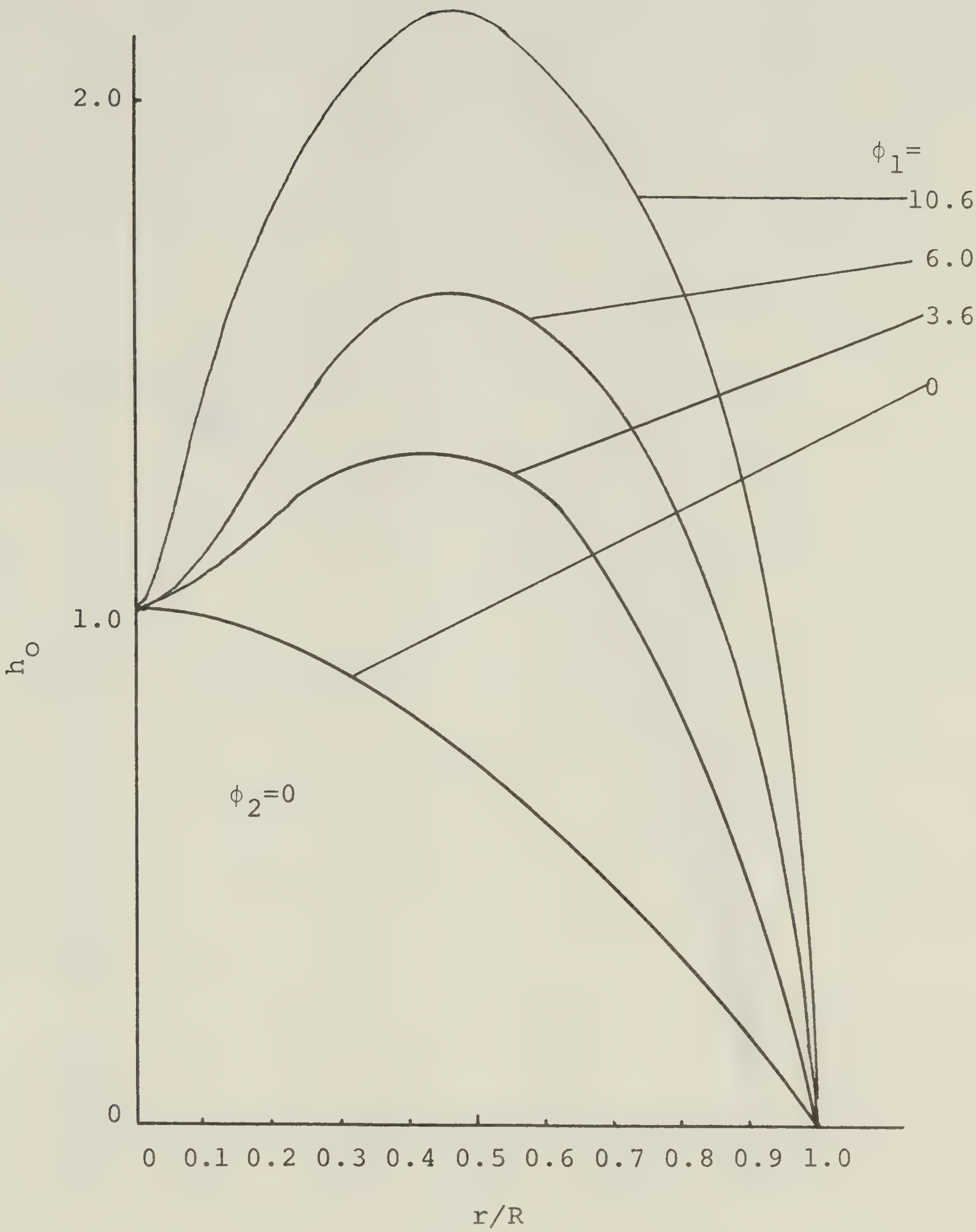

Fig. (6). The particle density vs radial position, in the presence of a finite m = 1 mode instability.

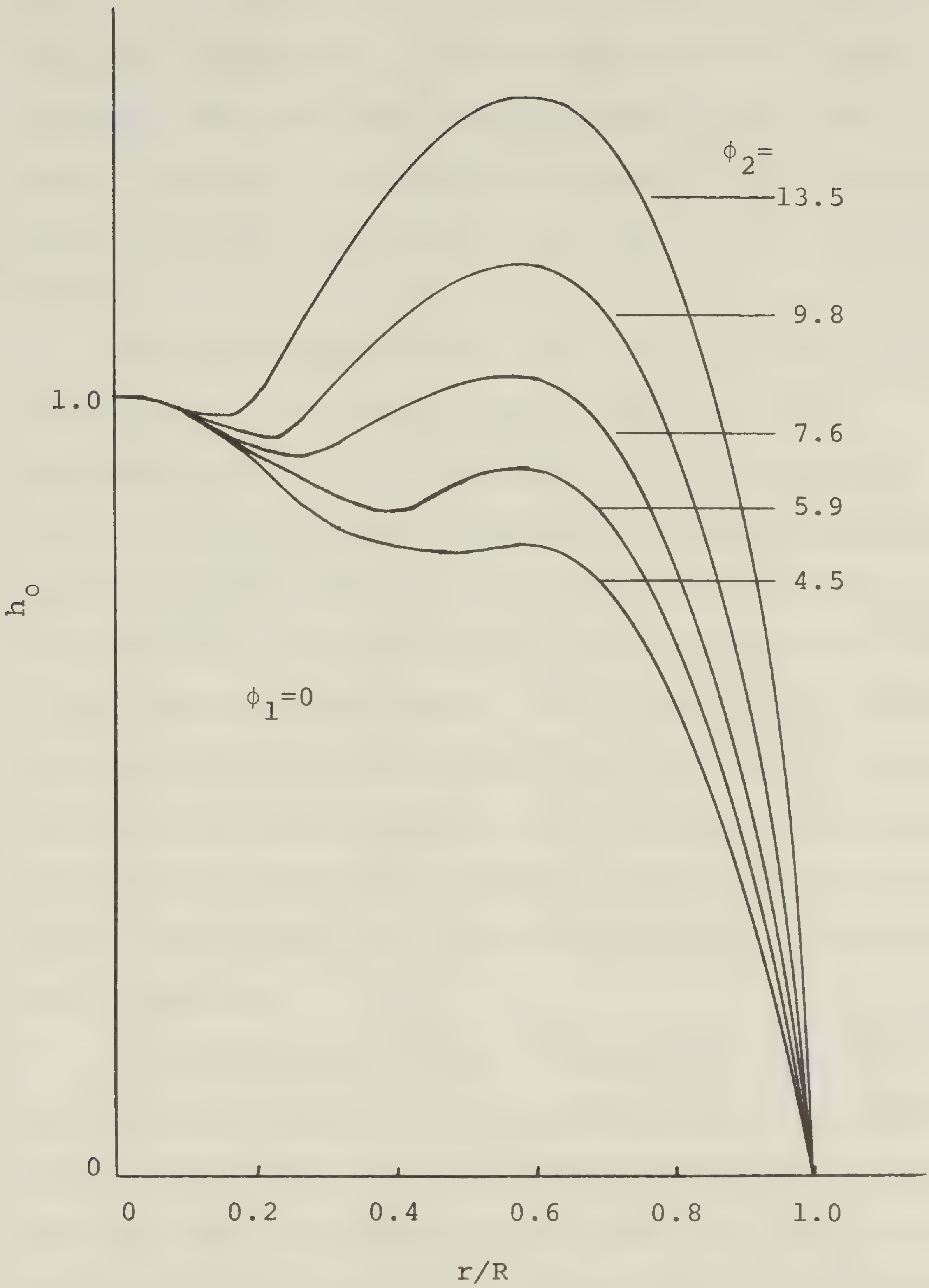

Fig.(7). The particle density vs radial position, in the presence of a finite m = 2 mode instability.

peak in the density occurs at approximately r/R = 0.60. For small values of ϕ_2 it was found that the radial function decreased more rapidly than a zero order Bessel function. Unlike the previous graph the profile shows an off axis dip before it rises to its maximum value.

The third graph (fig. (8)) shows the case of a constant $\phi_1 = 3.5$ and ϕ_2 varying from -6.0 to a maximum value of 22.7. (Looking at ϕ_2 in equation (II.60) we see that it is possible to have a negative ϕ_2 if the phase angle δ_2 is chosen such that sin δ_2 is negative). It can be seen from the graph that as ϕ_2 is increased positively, the off axis peak increases in magnitude and shifts toward the tube wall. As ϕ_2 is increased negatively, the off axis peak decreases in magnitude from the ϕ_2 equal to zero case and shifts toward the column axis. But no dip seems to occur near the column axis.

The last graph (fig. (9)) shows the case of a constant ϕ_2 fixed at a value of 20 while ϕ_1 is varied from -3.0 to 6.0. As ϕ_1 is increased positively the off axis peak increases in magnitude from the $\phi_1 = 0$ case and shifts towards the column axis. In addition a dip near the column axis becomes less pronounced and shifts towards the column axis, but is eventually

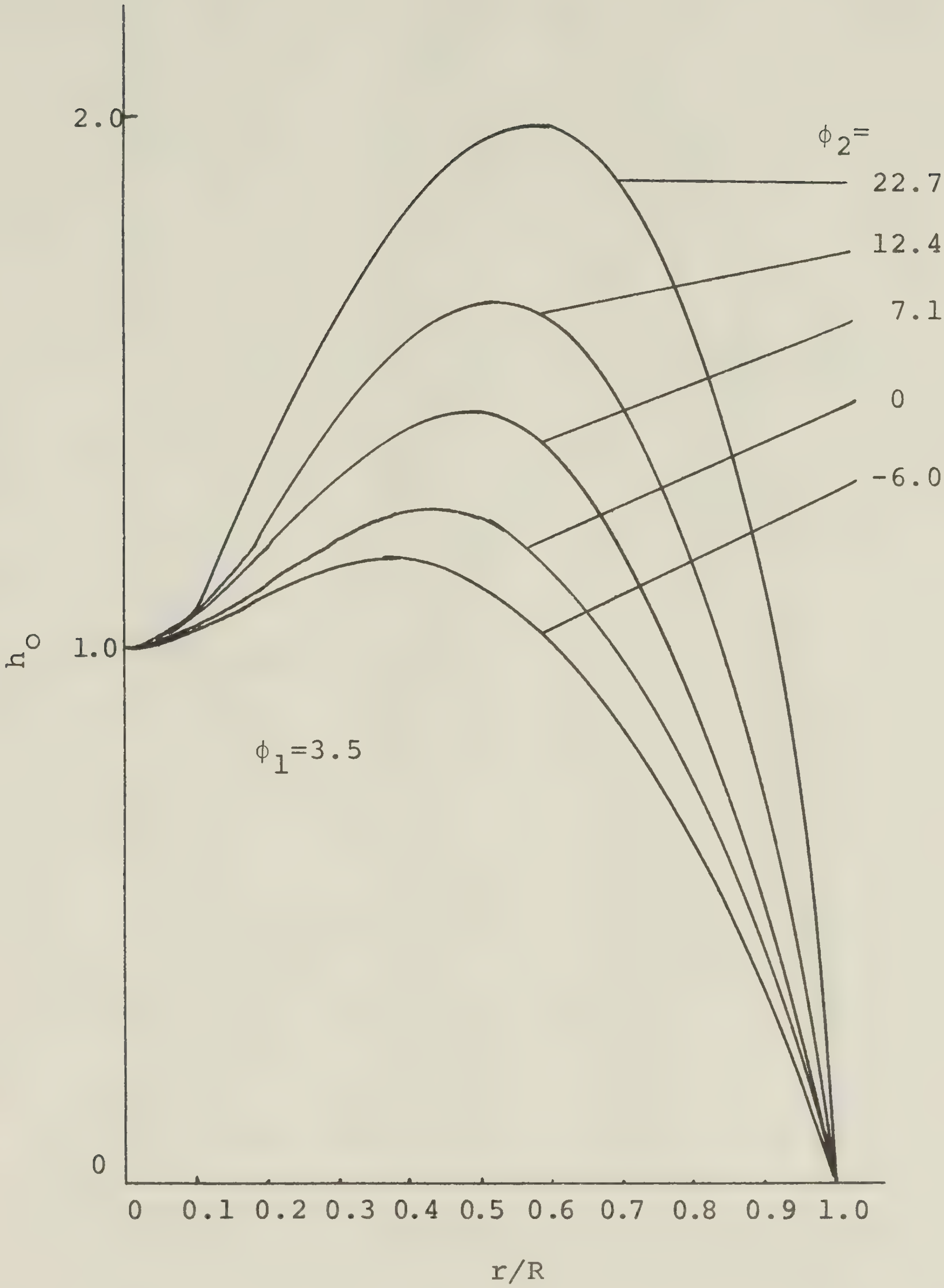

Fig.(8). The particle density vs radial position, in the presence of both the finite m= 1 and finite m = 2 modes of instabilities, for fixed m = 1 mode amplitude factor and variable m = 2 mode amplitude factor.

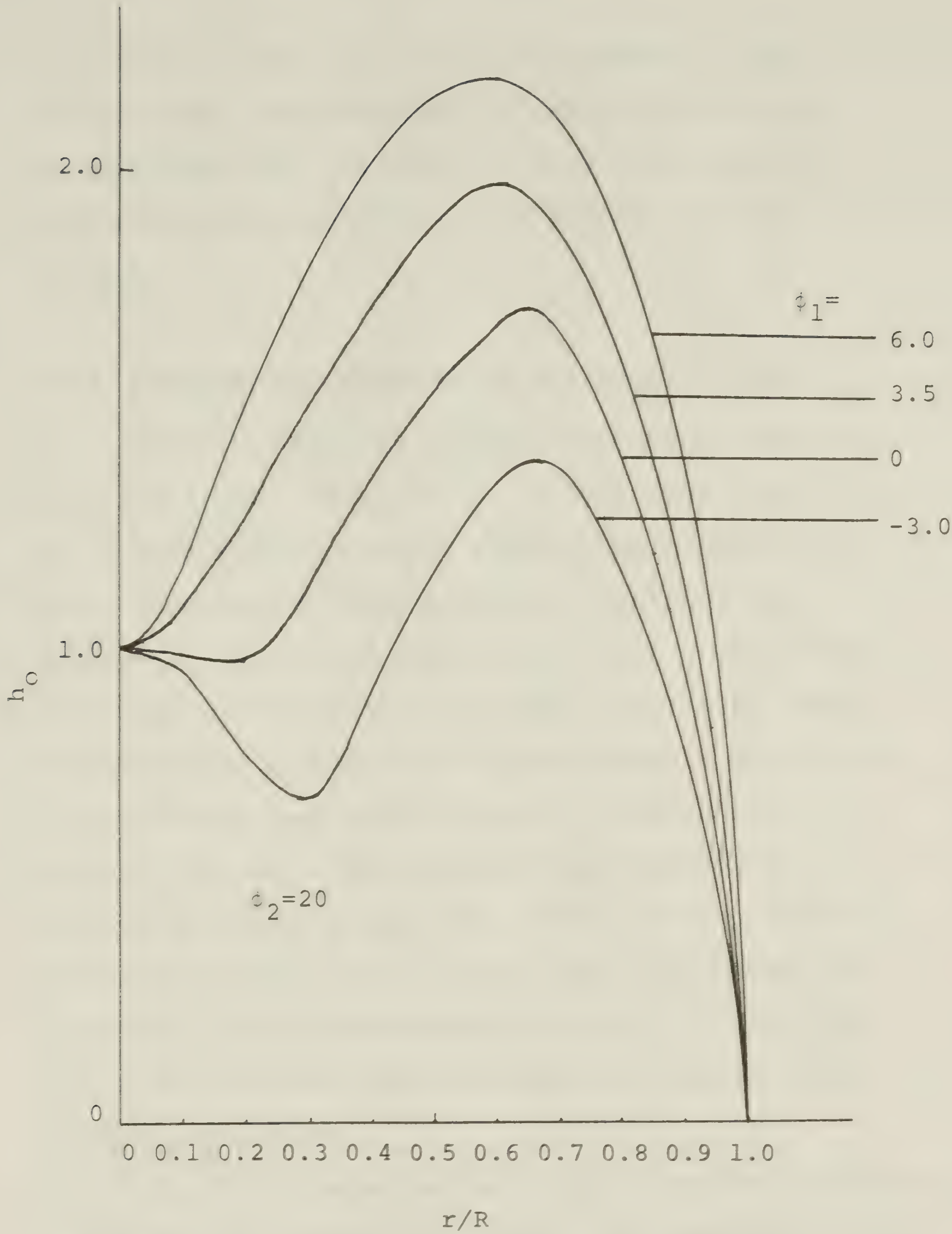

Fig.(9). The particle density vs radial position, in the presence of both the finite m = 1 mode and finite m = 2 mode instabilities, for fixed m = 2 amplitude factor and variable m = 1 amplitude factor.

removed for large ϕ_1. As ϕ_1 is increased negatively, the off axis peak decreases in magnitude and shifts towards the wall. The dip in the density becomes more pronounced and also shifts towards the wall of the tube.

II.4 Experimental Evidence of a Pure m = 2 Mode

The case which is of most interest is that of a pure m = 2 mode (fig. (7)). It does seem that a m = 2 mode could not occur without the presence of a fully developed m = 1 mode first. Ventrice and Massey[10] performed experiments using a very short wide tube with a length to radius ratio of 6. This differed from previous experiments where a long narrow tube had been used with a length to radius ratio greater than 400. The results they obtained were similar to those in fig. (7). Under the experimental conditions they used it appears that the conditions necessary for the development of the m = 1 mode could not be met whereas the conditions for the m = 2 mode could be satisfied.

One of the experimental graphs obtained by Ventrice and Massey is given in fig. (10). The resulting curves do resemble those in fig. (7) giving evidence for a fully developed m = 2 mode in the tube without a developed m = 1 mode. It can be seen from the graph

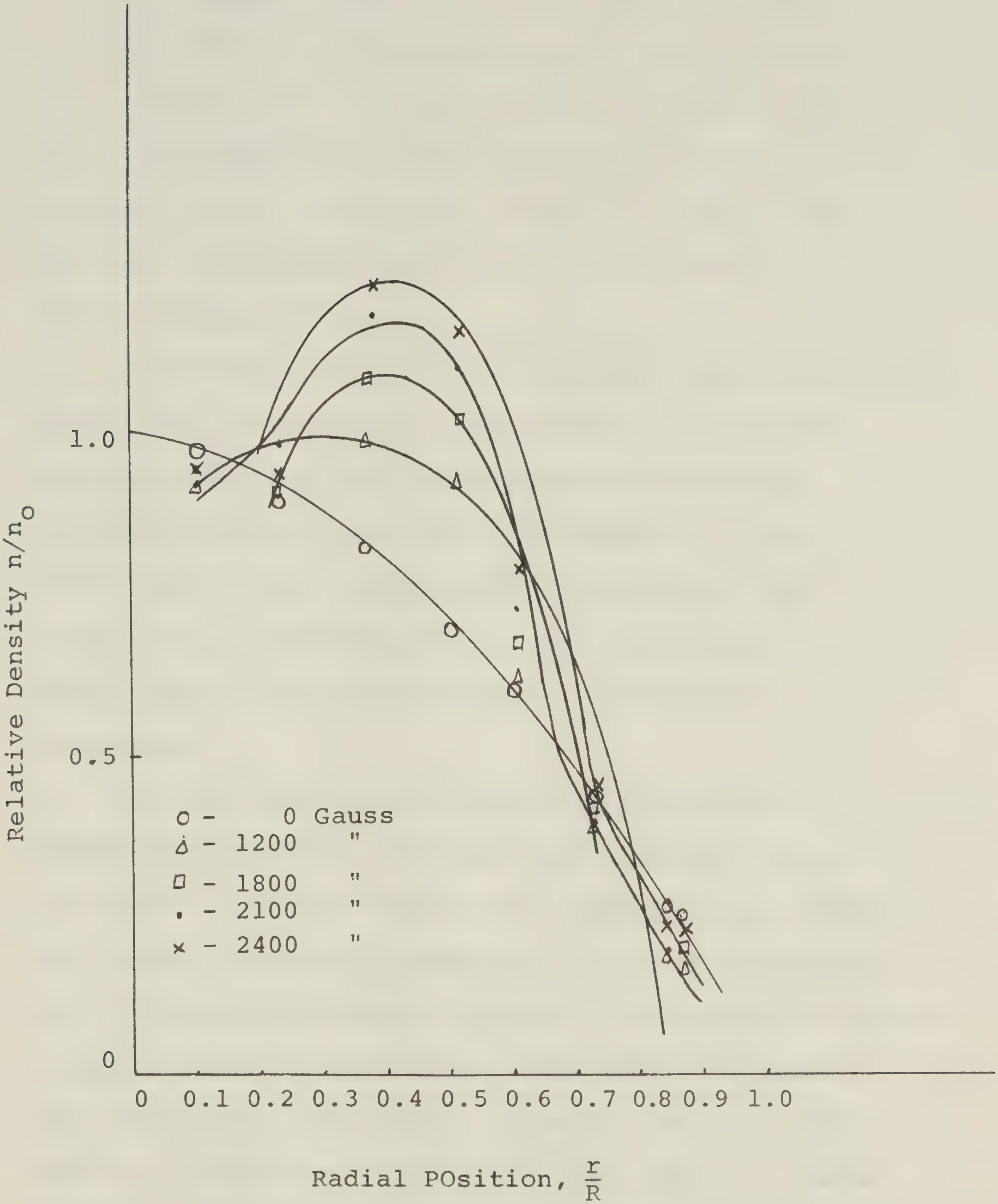

Fig.(10). The relative particle density (n/n_o) vs the radial position (r/R) for various values of the applied longitudinal magnetic field B_{oz} with a constant tube current of 100 mA.

that for larger values of B_{oz} the larger is the maximum value of the density curve. This is also in agreement with the curve in fig. (7) because as ϕ_2 is increased so is the maximum value of the density curve. Looking at equation (II.60) we can see that ϕ_2 depends linearly on B_{oz} so that as B_{oz} is increased so is ϕ_2.

Huchital and Holt[11,12] have pointed out in their paper that a transverse component of an externally applied magnetic field has a substantial effect upon the critical magnetic field strength. They even suggest that a large enough transverse component of the magnetic field could make the m = 2 mode rise in the tube before a m = 1 mode is developed.

It was with this in mind that they then performed an experiment using an applied axial electric field and a magnetic field with a large axial component on a small transverse component. It was observed that for a transverse magnetic field of 70.5 gauss and an axial magnetic field of 1880 gauss that the m = 1 mode was less stable than the m = 2 mode, that is to say the m = 1 mode grew more rapidly than the m = 2 mode. However, for a transverse magnetic field of 81.5 gauss and a longitudinal magnetic field of 1840 gauss the

$m = 2$ mode grew before the $m = 1$ mode. Therefore it does seem possible that if the applied magnetic field has a large enough component in the transverse direction it is possible for the $m = 2$ mode to grow in the tube before and independent of the $m = 1$ mode.

CHAPTER III

MODIFIED STEADY STATE SOLUTION

The purpose of the following calculation is to find under what conditions, if any, we can have a pure m = 2 mode growing in the tube before a m = 1 mode is developed. Ando and Glicksman[13] have pointed out that this situation may be possible if the azimuthal magnetic field is taken into consideration. In our calculation,therefore, we will take into consideration the effect of the azimuthal magnetic field. Also we will reduce the effect the walls have in causing the instability by considering a wide tube, instead of a long narrow tube as done by previous authors. We will also consider that most of the recombinations and generations occur in the bulk. The major factor affecting the radial density profile in the steady state solution is the induced azimuthal magnetic field instead of the effect of the walls as in the case of previous calculations.

III.1 Formulation of the Problem

We start our calculations from the same two basic equations as used by Johnson and Jerde. The first equation is a steady state force equation which relates the flux to the forces and can be written as

$$\vec{\Gamma}_{\mp} = \mu_{\mp} \left(- \frac{kT_{\mp}}{e} \nabla n \mp n\vec{E} \mp \vec{\Gamma}_{\mp} \times \vec{B}\right) \qquad \text{(III.1)}$$

"-" refers to the electrons and "+" refers to the ions, where n is the number density of ions or electrons, $\vec{\Gamma}_{\mp}$ are the particle flux, and are defined by $\vec{\Gamma}_{\mp} = n\vec{V}_{\mp}$, where $\vec{V}_{\mp}$ are the drift velocities. $D_{\mp} = \mu_{\mp} k_B T_{\mp}/e = \mu_{\mp} V_{\mp}$ are the diffusion constants, and $\mu_{\mp}$ are the mobilities. $T_{\mp}$ are the temperatures and $V_{\mp} = k_B T_{\mp}/e$ are the thermal energies expressed in electron volts. $\vec{E}$ and $\vec{B}$ are the actual electric and magnetic fields. Both $\vec{E}$ and $\vec{B}$ include the internal and external fields. The form for the fields in the steady state problem are as follows:

$$\begin{aligned} \vec{E} &= E_r(r)\hat{r} + E_{oz}\hat{z} \\ \vec{B} &= B_\phi(r)\hat{\phi} + B_{oz}\hat{z} \quad . \end{aligned} \qquad \text{(III.2)}$$

E_{oz} and B_{oz} are the externally applied fields, while $E_r(r)$ and $B_\phi(r)$ are the internal fields. $B_\phi(r)$ is the induced magnetic field in the azimuthal direction and is simply related to the electric current by the equation

$$B_\phi(r) = \frac{e}{c^2} \frac{1}{r} \int_0^r n(r')(V_z^+ - V_z^-)r'dr' \ , \qquad \text{(III.3)}$$

where $V_z^{\mp}$ are the Z components of the drift velocities.

Let us now examine the terms in equation (III.1). The first term is

$$- \frac{\mu_{\mp} kT_{\mp}}{e} \nabla n \ .$$

This is the flux of particles which arise from the density gradient and its direction is the same for both types of particles, although its magnitude is much larger for electrons than for ions. In the steady state case there is only a density gradient in the radial direction and therefore the flux of particles will be in towards the center.

The next term is $\mp \mu_{\mp} n\vec{E}$. This is the flux of particles due to the electric field. The applied electric field (III.2) causes the particles to move in opposite directions in the axial direction. However, the radial electric field acts in such a manner as to retard the flow of electrons to the wall while it enhances the flow of ions to the wall causing ambipolar diffusion.

The last term is $\mp \mu_{\mp}(\vec{\Gamma}_{\mp} \times \vec{B})$. This is the flux of particles which arises as a result of the Lorentz force of $(\vec{V}_{\mp} \times \vec{B})$, where $V_{\mp}$ is the velocity, due to the current and the magnetic field. This term causes

the electrons to gyrate about the B_{oz} lines in a right-handed sense while it causes the positive ions to rotate in the opposite direction. In the steady state case if you take the radial component of this expression you obtain

$$\mp \mu_{\mp}\hat{r}.(\vec{\Gamma}_{\mp} \times \vec{B}) = -(\mp \mu_{\mp}\Gamma^{z}_{\mp}B_{\phi}(r)) \quad , \qquad \text{(III.4)}$$

where $\Gamma^{z}_{\mp}$ is the flux of particles in the z direction. Since the electric field is applied parallel to the axial magnetic field we will assume that they are both applied in the positive axial direction. Therefore the ions will move in the positive z direction while the electrons will move in the negative z direction.

Now

$$\vec{\Gamma}_{\mp} = n\vec{V}_{\mp}$$

$$\therefore \quad \Gamma^{z}_{\mp} = \mp nV_{\mp} \quad .$$

Equation (III.4) can be written as

$$\mp \mu_{\mp}\hat{r}.(\vec{\Gamma}_{\mp} \times \vec{B}) = -\mu_{\mp}nV_{\mp}B_{\phi}(r) \quad . \qquad \text{(III.4a)}$$

It can be seen from equation (III.4a) that the effect of the self induced azimuthal magnetic field is to reduce the outward radial flow of particles and

therefore make the column more stable. If instead of taking the radial component of the expression we take the azimuthal component, we get

$$\mp \mu_{\mp}\hat{\phi}\cdot(\vec{\Gamma}_{\mp} \times \vec{B}) = \pm \mu_{\mp}\Gamma^{r}_{\mp}B_{oz} \ . \qquad \text{(III.4b)}$$

Now since the outward radial flow of particles is ambipolar $\Gamma^{r}_{\mp}$ is positive for both ions and electrons. Therefore it can be seen that the effect of B_{oz} is to cause the electrons to gyrate about the B_{oz} lines in a positive manner (right-handed sense) while it causes the ions to rotate in a negative direction.

Our second basic equation is the equation of continuity

$$\frac{\partial(n-\bar{n})}{\partial t} + \nabla\cdot\Gamma_{\mp} = (n-\bar{n})\xi \ , \qquad \text{(III.5)}$$

where $1/\xi$ is the mean bulk lifetime for excess carriers, and $\bar{n}$ is the density at which no net recombinations or generations take place.

Equation (III.5) states that the time change in the net number of particles available for recombination or generation in a small volume plus the divergence of the flux of particles is equal to the net number of recombinations or regenerations in that volume.

Using equations (III.1) we solve for $\vec{\Gamma}_{\mp}$, getting

$$\vec{\Gamma}_{\mp} = \mp \mu_{\mp}^{2}\mu_{\mp}^{'} n(\vec{B}.\vec{E})\vec{B} - \mu_{\mp}^{2} D_{\mp}^{'}\vec{B}(\vec{B}.\nabla n)$$

$$\mp \mu_{\mp}^{'} n\vec{E} - \mu_{\mp}\mu_{\mp}^{'} n(\vec{B} \times \vec{E}) - D_{\mp}\nabla n$$

$$\mp \mu_{\mp} D_{\mp}^{'}(\vec{B} \times \nabla n) \quad , \tag{III.6}$$

where

$$\mu_{\mp}^{'} = \frac{\mu_{\mp}}{1 + \mu_{\mp}^{2} B_{oz}^{2}} \; ; \quad D_{\mp}^{'} = \frac{D_{\mp}}{1 + \mu_{\mp}^{2} B_{oz}^{2}} \quad . \tag{III.7}$$

We next put our value for $\vec{\Gamma}_{\mp}$ into equation (III.5) and obtain for our continuity equation

$$- \frac{\partial}{\partial t}(n-\bar{n}) + (n-\bar{n})\xi = \nabla.[\mp \mu_{\mp}^{2}\mu_{\mp}^{'} n(\vec{B}.\vec{E})\vec{B}$$

$$- \mu_{\mp}^{2} D_{\mp}^{'}\vec{B}(\vec{B}.\nabla n) \mp \mu_{\mp}^{'} n\vec{E} - \mu_{\mp}\mu_{\mp}^{'} n(\vec{B} \times \vec{E})$$

$$- D_{\mp}^{'}\nabla n \mp \mu_{\mp} D_{\mp}^{'}(\vec{B} \times \nabla n)] \quad . \tag{III.8}$$

We have used cylindrical coordinates for reasons of symmetry in solving for $\vec{\Gamma}_{\mp}$. The z axis is taken along the direction of applied electric and magnetic

fields. We have neglected $[B_\phi(r)/B_{oz}]^2$ in comparison with unity. Here $B_\phi(r)$ is the self induced azimuthal magnetic field.

III.2 Steady State Solution

In the steady state solution there are no variations in the density with time. Therefore $\partial(n-\bar{n})/\partial t=0$. In view of the symmetry and our choice of cylindrical coordinate the density distribution will be in the radial direction, therefore we need only consider ourselves with the radial component of equation (III.8).

Therefore using our steady state condition and remembering that $\nabla.B=0$, only the r dependent terms from equation (III.8) remain

$$
\begin{aligned}
(n-\bar{n})\xi = &\mp \mu'_{\mp} E_r \frac{\partial n}{\partial r} \mp \mu'_{\mp} n \frac{1}{r} \frac{\partial}{\partial r} (rE_r) \\
&- \mu_{\mp}\mu'_{\mp} \frac{n}{r} B_\phi(r) E_{oz} \mp \mu_{\mp}\mu'_{\mp} \frac{n\partial B_\phi(r)}{\partial r} E_{oz} \\
&- \mu_{\mp}\mu'_{\mp} B_\phi(r) E_{oz} \frac{\partial n}{\partial r} - D'_{\mp} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial n}{\partial r}\right) . \qquad \text{(III.9)}
\end{aligned}
$$

Using equations (III.9) we eliminate the radial electric field $E_r(r)$ and obtain one equation for $n(r)$ in terms of the induced magnetic field $B_\phi(r)$ and other constant terms.

$$(\mu_-' + \mu_+')(n-\bar{n})\xi = -\frac{n}{r} B_\phi E_{oz}(\mu_-' + \mu_+')(\mu_- + \mu_+)$$

$$- n \frac{dB_\phi}{dr} E_{oz}\mu_-'\mu_+'(\mu_- + \mu_+) - B_\phi E_{oz} \frac{dn}{dr} \mu_-'\mu_+'(\mu_- + \mu_+)$$

$$- \frac{1}{r}\frac{d}{dr}\left(r\frac{dn}{dr}\right)(D_-'\mu_+' + D_+'\mu_-') \quad . \qquad \text{(III.10)}$$

Using definitions (III.7) and $D_\mp = \mu_\pm kT_\pm/e$ and dividing through by $(\mu_- + \mu_+)$ equation (III.10) becomes

$$\left(\frac{1}{\mu_+} + \frac{1}{\mu_-}\right)^{-1} \frac{1}{r}\frac{d}{dr}\{r[B_\phi n(\mu_+ + \mu_-)E_{oz} + \frac{k_B}{e}(T_+ + T_-)\frac{dn}{dr}]$$

$$+ (n-\bar{n})(1 + \mu_- {}_+B_{oz}^2)\xi\} = 0 \quad . \qquad \text{(III.11)}$$

Multiplying both sides of equation (III.11) by rdr and integrating from 0 to r, and also using equation (III.3) for $B_\phi(r)$, we have

$$\frac{dn}{dr} = -\frac{E_{oz}n(r)e^2(\mu_- + \mu_+)(V_{+z} - V_{-z})}{c^2k_B(T_+ + T_-)}\int_0^r n(r')r'dr'$$

$$+ \frac{e}{k_B(T_+ + T_-)}\left(\frac{1}{\mu_+} + \frac{1}{\mu_-}\right)\frac{1}{r}\{\gamma\int_0^r [n(r')-\bar{n}]r'dr'\}$$

(III.12)

where

$$\gamma = -\xi(1 + \mu_-\mu_+B_{oz}^2) \quad . \qquad \text{(III.13)}$$

Defining the following dimensionless quantities

$$x = \frac{r}{R} \qquad q(r) = \frac{n(r)}{n_a}$$

$$\Gamma = \frac{e^2R^2(V_{+z} - V_{-z})(\mu_- + \mu_+)n_aE_{oz}}{c^2k_B(T_- + T_+)} \tag{III.14}$$

$$\alpha_1 = \gamma\left(\frac{1}{\mu_+} + \frac{1}{\mu_-}\right) \frac{eR^2}{k_B(T_+ + T_-)} ,$$

where R is the radius of the tube and n_a is the average density of the plasma. Using equations (III.14) equation (III.11) can be rewritten as

$$\frac{dq}{dx} = - \frac{\Gamma q}{x} \int_0^x q(r)tdt + \frac{\alpha_1}{x} \int_0^x (q(t) - \eta)tdt \tag{III.15}$$

where

$$\eta = \frac{\bar{n}}{n_a} .$$

Equation (III.15) is a non-linear equation and cannot be solved very easily. However, it has been solved by Bennett[14] when $\alpha_1 = 0$. The solution obtained is

$$q_B(x) = \frac{Q_o}{\left[1 + \left(\frac{x}{x_o}\right)^2\right]^2} , \tag{III.16}$$

where $x_o^2 Q_o = 8/\Gamma$, and $x/x_o < 1$, and Q_o is the plasma density at the origin. In putting α_1 equal to zero Bennett has neglected the recombinations and generations which occur in the bulk of the plasma. We attempt to include the effect of recombinations and generations in an approximate manner.

Let us assume that the term

$$\frac{\alpha_1}{x} \int_0^x (q - \eta)\ tdt$$

in equation (III.15) is small. This is reasonable if α_1 is small. We will consider the solution to equation (III.15) to be a perturbation of the Bennett solution, (equation (III.16)). We write the solution as

$$q(x) = q_B(x) + \chi(x) \quad , \tag{III.17}$$

where $\chi(x)$ is the perturbed term. We let $\chi(x)$ be proportional to α_1 and expand the coefficient of α_1 in a power series of x, keeping only the first five terms. Therefore $\chi(x)$ can be written as

$$\chi(x) = \alpha_1 A_o + \alpha_1 A_1 x + \alpha_1 A_2 x^2 + \alpha_1 A_3 x^3 + \alpha_1 A_4 x^4 \quad , \tag{III.18}$$

where the coefficients A_o, A_1, A_2, A_3, A_4 have to be determined.

To determine the coefficients in the expansion for $\chi(x)$ we substitute (III.17) into (III.15) using (III.16) and (III.18), keeping only the lowest order terms in α_1 (i.e. neglecting terms of order $(\alpha_1)^2$). Equation (III.15) becomes

$$A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 = -\frac{\Gamma Q_o}{x[1 + (\frac{x}{x_o})^2]^2}$$

$$[\frac{A_ox^2}{2} + \frac{A_1x^3}{3} + \frac{A_2x^4}{4} + \frac{A_3x^5}{5} + \frac{A_4x^6}{6}]$$

$$- \frac{\Gamma}{x}[A_o + A_1x + A_2x^2 + A_3x^3 + A_4x^4] \frac{4x^2}{\Gamma x_o^2[1 + (\frac{x}{x_o})^2]}$$

$$+ \frac{4x}{\Gamma x_o^2[1 + (\frac{x}{x_o})^2]} - \frac{\eta}{4}x^2 \quad . \qquad \text{(III.19)}$$

Using equation (III.19) and equating the coefficients of like powers of x equal to zero, we can solve for the coefficients in the expansion for $\chi(x)$. In determining the coefficients we have used the binominal expansion, since we have assumed $(x/x_o) < 1$, and the equation $1/x_o^2 = Q_o\Gamma/8$ in equation (III.19).

It follows that

$$A_1 = A_3 = 0 \tag{III.20}$$

$$\left.\begin{aligned} A_2 &= -\frac{\Gamma Q_o}{2} A_o + \frac{1}{4}(Q_o - \eta) \\ A_4 &= \frac{9}{64}(\Gamma Q_o)^2 A_o - \frac{1}{16}(\Gamma Q_o)Q_o + \frac{3}{64}(\Gamma Q_o)\eta. \end{aligned}\right\} \tag{III.21}$$

The coefficient A_o cannot be determined from equation (III.19), but will be determined below.

III.3 Calculation of A_o

To calculate A_o we write down the equation which states that the total number of generations inside the plasma is equal to the total number of recombinations at the walls. However, since we are considering a wide tube, as pointed out earlier, the effects of the walls on the density in the bulk will be small. The equation is

$$\int_0^R \xi(n(r)-\bar{n})rdr = RS(n(r)-\bar{\bar{n}}) \ , \tag{III.22}$$

where S is the surface recombination velocity and $\bar{\bar{n}}$ is the density at the wall at which no net recombinations or generations take place.

It should be pointed out that although equation (III.22) is an exact equation our solution for $\alpha_1 A_o$ will be crude. The reason for this is that we keep terms up to powers of x^4 in our density distribution. Therefore a truly exact value for $\alpha_1 A_o$ is not obtained. However, our calculated value will give us an approximate correction to the Bennett solution on and near the axis.

It can be shown by physical arguments that $\alpha_1 A_o$ must be negative. The generations inside the bulk will occur when the density of the plasma is less than $\bar{n}$. Now since the Bennett solution is a maximum at the axis and a minimum at the walls, and since we insist on a net number of generations in the bulk, these generations will have to occur away from the tube axis. However, the Bennett plasma density along the axis, Q_o, is greater than $\bar{n}$ therefore along the axis there will be net recombinations and the plasma density will decrease from the Bennett value, Q_o. Therefore we would expect from physical arguments that the first term in our density perturbation to be negative.

Rewriting equation (III.22) in terms of variable x and doing the required integration keeping only terms up to $\alpha_1 A_4$, we obtain a very approximate equation which can be solved for $\alpha_1 A_o$

$$\frac{Q_o}{2[1 + (\frac{1}{x_o})^2]} + \frac{\alpha_1 A_o}{2} + \frac{\alpha_1 A_2}{4} + \frac{\alpha_1 A_4}{6} + \frac{\eta}{2}$$

$$\approx \rho[\frac{Q_o}{[1 + (\frac{1}{x_o})]} + \alpha_1 A_o + \alpha_1 A_2 + \alpha_1 A_4 - \mu] \qquad \text{(III.23)}$$

where

$$\mu = \frac{\bar{\bar{n}}}{n_a}$$

$$\rho = \frac{S}{R\xi} \quad .$$

In obtaining equation (III.23) we have used the fact that $A_1 = A_3 = 0$.

We rewrite equation (III.23) expanding the term $[1 + (1/x_o)^2]^{-1}$ in a binominal expansion keeping terms up to $(1/x_o)^4$. This expansion is allowed since $1/x_o < 1$. Also, we make substitutions into equation (III.23) for the coefficients $\alpha_1 A_2$, $\alpha_1 A_4$ using equations (III.2).

Equation (III.23) becomes

$$\alpha_1 A_o(\frac{1}{2} - \frac{\Gamma Q_o}{8} + \frac{3}{128}(\Gamma Q_o)^2 - \rho(1 - \frac{\Gamma Q_o}{2} + \frac{9}{64}(\Gamma Q_o)^2)$$

$$\approx \rho(Q_o(1 - \frac{\Gamma Q_o}{4}) + \frac{\alpha_1}{4}(Q_o - \eta) - \frac{\alpha_1}{16}(\Gamma Q_o)Q_o$$

$$+ \frac{3\alpha_1}{64}(\Gamma Q_o)\eta - \mu) - (\frac{Q_o}{2}(1 - \frac{\Gamma Q_o}{8})$$

$$+ \frac{\alpha_1}{16}(Q_o - \eta) - \frac{\alpha_1}{96}(\Gamma Q_o)Q_o + \frac{\alpha_1}{128}(\Gamma Q_o)\eta - \frac{\eta}{2}) . \qquad \text{(III.24)}$$

We will keep only the dominant terms in equation (III.24). Since we are considering the effect of the walls to be small we can consider ρ to be small, such that we can neglect terms of order ρ, $\rho|\alpha_1|$ relative to unity. We also neglect $|\alpha_1|(\Gamma)$ relative to unity.

Therefore equation (III.24) can be written as

$$\alpha_1 A_o \approx -(Q_o - \eta) - \frac{\Gamma Q_o}{4}\left(\frac{Q_o}{2} - \eta\right) + \frac{|\alpha_1|}{8}(Q_o - \eta)$$

$$+ \rho\left(Q_o - \frac{\eta}{2} - 2\mu\right) \quad . \tag{III.25}$$

From equation (III.25) we can see that $\alpha_1 A_o$ is negative as long as $Q_o > 2\eta$. Therefore even though our calculation for $\alpha_1 A_o$ is crude it is negative as we expected by our physical argument. Since $\alpha_1 A_o$ is negative it can be seen by looking at equation (III.21) that $\alpha_1 A_2$ is positive and $\alpha_1 A_4$ is negative.

We can rewrite the radial density equation (III.15) using equations (III.25) and (III.21) remembering that our solution now is very approximate. Equation (III.15) becomes

$$q(x) \approx Q_o\left(1 - \frac{\Gamma Q_o}{4}x^2\right) - (Q_o - \eta) - \frac{\Gamma Q_o}{4}\left(\frac{Q_o}{2} - \eta\right)$$

$$+ \frac{|\alpha_1|}{8}(Q_o - \eta) + \rho\left(Q_o - \frac{\eta}{2} - 2\mu\right)$$

$$+ \left(\frac{\Gamma Q_o}{2}(Q_o - \eta) - \frac{|\alpha_1|}{4}(Q_o - \eta)\right)x^2 \quad . \tag{III.26}$$

In obtaining (III.26) we have expanded the Bennett solution (equation (III.10)) in a binominal expansion and throughout the equation we kept terms only up to a power of x^2.

III.4 Calculation of the Radial Component of the Internal Electric Field

The radial component of the internal electric field E_r can be determined from the condition that in the steady state the radial flux of both type of charges must be equal

$$\Gamma_-^r = \Gamma_+^r \quad . \tag{III.27}$$

Now, it can be seen from equation (III.6) that

$$\Gamma_{\mp}^r = \mp \mu_{\mp}' n E_r - \mu_{\mp}\mu_{\mp}' n B_\phi(r) E_{oz} - D_{\mp}' \frac{\partial n}{\partial r} \quad . \tag{III.28}$$

Therefore using equations (III.27) and (III.28) we obtain for the radial component of the internal electric field

$$E_r = E_{or}(r) - aB_\phi(r)E_{oz} \quad , \tag{III.29}$$

where

$$E_{or} = \frac{\alpha}{n}\frac{dn}{dr} \quad , \quad \alpha = \frac{D_+' - D_-'}{\mu_+' + \mu_-'} \tag{III.30}$$

$$a = \frac{\mu_- - \mu_+}{1 + \mu_-\mu_+ B_{oz}^2} \quad .$$

Now rewriting equation (III.29) in terms of the variable x we obtain

$$E_r(x) = E_{or}(x) - aB_\phi(x)E_{oz} \quad , \tag{III.31}$$

where

$$E_{or}(x) = \alpha \frac{1}{q}\frac{dq}{dx} \tag{III.32}$$

$$\alpha = \frac{1}{R}\,\frac{D_+^{'} - D_-^{'}}{\mu_+^{'} + \mu_-^{'}} \quad ,$$

a is the same as in (III.30).

CHAPTER IV

DISCUSSION OF RESULTS

Equation (III.26) can be rewritten keeping only the most dominant terms as

$$q(x) \approx Q_o(1 - \frac{\Gamma Q_o}{4} x^2) - (Q_o - \eta) + \frac{\Gamma Q_o}{2}(Q_o - \eta)x^2. \qquad \text{(IV.1)}$$

This can be rewritten as

$$q(x) \approx \eta + \frac{\Gamma Q_o}{2} (\frac{Q_o}{2} - \eta) x^2 . \qquad \text{(IV.2)}$$

It can be seen by looking at equation (IV.2) that the effect of the perturbed density term is to reduce the density at the cylindrical axis ($x = 0$) from the Bennett value of Q_o which is greater than 2 to a different value η which is greater than 1.

For the region near the cylindrical axis the density will increase quadratically provided $Q_o > 2\eta$, which we have assumed. However, it should not be assumed that the density profile will continue to increase quadratically as x becomes larger. In fact it should not be assumed that the density profile will continue to increase at all for larger values of x. As x increases the effect of the terms which

contain the higher powers of x will dominate the solution. The effect of these terms has not been kept in our approximate equation (III.26). In fact it is expected that the higher power terms will dominate over the quadratic terms and cause the radial density to decrease to a value close to the density at the wall given by the Bennett solution. Therefore, a density profile similar to that obtained by Daugherty and Ventrice (fig. (7)) will be obtained except that the density at the wall will not be zero. However, it is difficult to determine exactly how many terms in the series must be kept before this is possible.

Therefore, all that can be said about our approximate solution is that it can give a radial form similar to the graph obtained by Daugherty and Ventrice (fig. (7)) for small values of r. For a pure m = 2 mode to fully develop in the tube before a m = 1 mode a form very similar to ours is needed. To obtain a complete radial density profile for our problem one would have to include much higher powers of x in our solution for $\alpha_1 A_o$ and throughout the paper. This could not be done very easily analytically but could probably be done by computer.

BIBLIOGRAPHY

1. Bohm, D., Burhop, E.H.S., Massey, H.S.W. and Williams, R.M., The Characteristics of Electrical Discharges in Magnetic Fields, pp. 62, 201, 49, 14, 31.
2. Lehnert, B., in Proceedings of the Second United Nations International Conference on Peaceful Uses of Atomic Energy, (United Nations, Geneva, 1958), Vol. 32, p.349.
3. Kadomtsev, B.B., and Nedospasov, A.V., Jour. Nuclear Energy C1 (1960) 230.
4. Hoh, F.C., Lehnert, B., Phys. Rev. Letters 7 (1961) 75.
5. Johnson, R.R. and Jerde, D.A., Phys. Fluids 5 (1962) 988.
6. Paranjape, B.V., Seale, D., Ng, K.C. and Johnson, R.R., Phys. Fluids 12 (1969) 1865.
7. Holter, O. and Johnson,R.R., Phys. Fluids 8 (1965) 333.
8. Holder, O. and Johnson, R.R., Phys. Fluids 9 (1966) 622.
9. Daugherty, T.L. and Ventrice, C.A., Phys. Fluids 14 (1971) 713.
10. Ventrice, C.A. and Massey, C.G., Phys. Fluids 11 (1968) 1990.

11. Huchital, D.A. and Holt, E.H., Phys. Rev. Letters 16 (1966) 677.

12. Huchital, D.A. and Holt, E.H., Appl. Phys. Letters 8 (1966) 321.

13. Ando, K. and Glicksman, M., Phys. Rev. 154 (1967) 316.

14. Bennett, W.H., Phys. Rev. 45 (1934) 890.